LINEAR DISJOINTNESS AND VALUES OF MAHLER M-FUNCTIONS

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Abstract. We prove that the algebra generated by the values of Mahler M-functions at algebraic points admits a decomposition as the tensor product, over $\overline{\mathbb{Q}}$, of a countable family of linearly disjoint algebras. Consequently, the study of linear and algebraic relations among such values reduces to the corresponding problem within each of these smaller algebras. Our proof relies on recent advances of the authors on Mahler's method in several variables. As an application, we obtain a complete description of the algebraic relations among the values of the Fredholm series and their derivatives at nonzero algebraic points. This fully resolves an old problem originating in Mahler's work.

1. Introduction

A recurrent strategy throughout mathematics consists in decomposing a complicated problem into simpler subproblems and then recovering the solution of the original problem from the solutions of its components. In representation theory, for instance, one seeks to decompose a representation into irreducible representations. In dynamical systems, one studies invariant subsets and, whenever possible, decomposes the system into ergodic components. Spectral decompositions provide another familiar manifestation of the same principle. In all these situations, the global structure can be recovered from suitably chosen elementary pieces.

Questions of linear or algebraic independence are, however, intrinsically global, which prevents such a strategy from working in general. To illustrate this elementary obstruction, let $x_1, \dots, x_n$ be linearly (resp. algebraically) independent over a field $\mathbb{K}$. Then every proper subset of

$$\{x_1, \dots, x_n, x_1 + \cdots + x_n\}$$

is linearly (resp. algebraically) independent over $\mathbb{K}$, whereas the whole set is not. Thus, even a complete understanding of the relations within all proper subfamilies does not determine the relations within the entire family.

In this paper, we exhibit what seems to be the first general unconditional setting in transcendental number theory in which the global obstruction described above can nevertheless be overcome. More precisely, we prove that the $\overline{\mathbb{Q}}$-algebra $\mathbf{M}$ generated by all values of Mahler M-functions at algebraic points admits a decomposition as the tensor product, over $\overline{\mathbb{Q}}$, of a countable

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family of linearly disjoint algebras. Consequently, the study of all linear and algebraic relations among arbitrary M-values reduces to the corresponding problem within each of these smaller algebras. This class of numbers has been studied since the pioneering work of Mahler and has attracted renewed interest in recent years, notably because of its close connections with finite automata and automatic numbers, as well as with a conjecture of Furstenberg concerning expansions of real numbers in multiplicatively independent bases; see [5, 6].

Let $\mathbb{K} \subset \mathbb{L}$ be fields, and let $\mathbb{E}$ and $\mathbb{F}$ be two $\mathbb{K}$-subalgebras of $\mathbb{L}$. They are said to be *linearly disjoint over* $\mathbb{K}$ if the canonical map

$$\begin{array}{rcl} \mathbb{E} \otimes_{\mathbb{K}} \mathbb{F} & \longrightarrow & \mathbb{L}, \\ \xi \otimes \eta & \longmapsto & \xi\eta, \end{array}$$

is injective. Equivalently, every finite family of elements of $\mathbb{E}$ that is linearly independent over $\mathbb{K}$ remains linearly independent over $\mathbb{F}$. Similarly, we say that $\mathbb{E}$ and $\mathbb{F}$ are *free over* $\mathbb{K}$ if every finite family of elements of $\mathbb{E}$ that is algebraically independent over $\mathbb{K}$ remains algebraically independent over $\mathbb{F}$. When $\mathbb{K}$ is algebraically closed and has characteristic zero, linear disjointness and freeness are equivalent. These notions extend naturally from two algebras to an arbitrary family. Let $\mathbb{K} \subset \mathbb{L}$ be fields, and let $(\mathbb{E}_i)_{i\in I}$ be a family of $\mathbb{K}$-subalgebras of $\mathbb{L}$. We say that the algebras $\mathbb{E}_i$, $i \in I$, are linearly disjoint (resp. free) over $\mathbb{K}$ if, for every $i_0 \in I$, the algebras

$$\mathbb{E}_{i_0} \quad \text{and} \quad \mathbb{K}[\mathbb{E}_i : i \in I \setminus \{i_0\}]$$

are linearly disjoint (resp. free) over $\mathbb{K}$.

Linear disjointness therefore provides precisely the decomposition principle that is missing for arbitrary families of numbers: once the relations within each individual algebra are known, all global relations are obtained by combining in a trivial way the relations within each subalgebra.

It is a fundamental notion in field theory and Galois theory. Variants of the same principle also play an important role in differential and difference Galois theory, as well as in the model-theoretic study of independence. By contrast, its explicit use in transcendental number theory seems to be extremely scarce. The main examples we could find occur in [11, 30, 8]. They are conditional results relying on Schanuel's conjecture, the Grothendieck period conjecture, or André's generalization of the latter, and were developed following a remark of Lang [16, pp. 30–31]. For instance, assuming Schanuel's conjecture, the authors of [11] proved that

$$\overline{\mathbb{Q}}[e, e^e, e^{e^e}, e^{e^{e^e}}, \ldots] \quad \text{and} \quad \overline{\mathbb{Q}}[\pi, \log(\pi), \log(\log(\pi)), \log(\log(\log(\pi))), \ldots]$$

are linearly disjoint over $\overline{\mathbb{Q}}$, where, at each stage, log denotes a suitable determination of the logarithm.

1.1. **Linear disjointness for M-values.** We now state our main results more precisely. Throughout the paper, we let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, and $\overline{\mathbb{Q}}^\star = \overline{\mathbb{Q}} \setminus \{0\}$.

Let $q \geq 2$ be an integer. An *M_q-function* is a power series $f \in \overline{\mathbb{Q}}[[z]]$ for which there exist polynomials $a_0, \ldots, a_m \in \overline{\mathbb{Q}}[z]$, not all zero, such that

$$a_0(z)f(z) + a_1(z)f(z^q) + \cdots + a_m(z)f(z^{q^m}) = 0. \tag{1.1}$$

An *M-function* is a power series that is an M_q-function for some integer $q \geq 2$. We recall that an M-function is either rational or transcendental over $\overline{\mathbb{Q}}(z)$. In either case, it is analytic in a neighborhood of the origin and meromorphic in the open unit disk. In the transcendental case, the unit circle is a natural boundary. Consequently, such a function can only be evaluated at points of the open unit disk $\mathcal{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ that are not poles.

Our main object of interest is the $\overline{\mathbb{Q}}$-algebra

$$\mathbf{M} := \overline{\mathbb{Q}}\big[f(\alpha) : \alpha \in \overline{\mathbb{Q}}^\star \cap \mathcal{D},\ f \text{ is an } M\text{-function well-defined at } \alpha\big].$$

Let $\mathbb{K} \subset \overline{\mathbb{Q}}$ be a field, let $q \geq 2$, and let $\alpha \in \mathbb{K}$ satisfy $0 < |\alpha| < 1$. We set

$$\mathbf{M}_{\mathbb{K},q,\alpha} := \big\{f(\alpha) : f \in \mathbb{K}[[z]] \text{ is an } M_q\text{-function well-defined at } \alpha\big\}.$$

This defines a family of $\mathbb{K}$-algebras.

Recall that two nonzero complex numbers θ and η are *multiplicatively dependent* if $\theta^u = \eta^v$ for some nonzero pair of integers (u, v). Given (q, α) and (r, β) in $\mathbb{Z}_{\geq 2} \times \overline{\mathbb{Q}}^\star$, we write $(q, \alpha) \sim (r, \beta)$ if q and r are multiplicatively dependent and α and β are multiplicatively dependent. This defines an equivalence relation.

Our main result reads as follows.

Theorem 1.1. *Let $\mathbb{K} \subset \overline{\mathbb{Q}}$ be a field. Then the following assertions hold.*

(a) *Let $(q, \alpha), (r, \beta) \in \mathbb{Z}_{\geq 2} \times (\mathbb{K}^\star \cap \mathcal{D})$. Then*

$$\mathbf{M}_{\mathbb{K},q,\alpha} = \mathbf{M}_{\mathbb{K},r,\beta} \iff (q, \alpha) \sim (r, \beta).$$

(b) *Let $\mathcal{I} \subset \mathbb{Z}_{\geq 2} \times (\mathbb{K}^\star \cap \mathcal{D})$ be such that $(q, \alpha) \not\sim (r, \beta)$ for every pair of distinct elements $(q, \alpha), (r, \beta) \in \mathcal{I}$. Then the family of $\mathbb{K}$-algebras $\mathbf{M}_{\mathbb{K},q,\alpha}$, $(q, \alpha) \in \mathcal{I}$, is linearly disjoint over $\mathbb{K}$.*

Remark 1.2. One of the main motivations for [5] was to prove, in connection with Furstenberg's dimension conjecture (see the discussion in [5, Sec. 2]), that an irrational real number cannot be generated by finite automata in two multiplicatively independent integer bases [5, Thm. 2.2]. This result follows immediately from Theorem 1.1. Indeed, suppose that a real number ξ admits expansions in the multiplicatively independent integer bases b_1 and b_2 whose digit sequences are generated by finite automata with respective input bases q_1 and q_2. Then

$$\xi \in \mathbf{M}_{\mathbb{Q},q_1,1/b_1} \cap \mathbf{M}_{\mathbb{Q},q_2,1/b_2}.$$

Since b_1 and b_2 are multiplicatively independent, the pairs $(q_1, 1/b_1)$ and $(q_2, 1/b_2)$ are not equivalent. Hence, by Theorem 1.1, the two algebras above are linearly disjoint over $\mathbb{Q}$, and their intersection is therefore equal to $\mathbb{Q}$. It follows that ξ is rational.

Theorem 1.1 yields the structural decomposition of the $\overline{\mathbb{Q}}$-algebra $\mathbf{M}$ announced above.

Corollary 1.3. *Let $\mathcal{R}$ be a set of representatives of $\big(\mathbb{Z}_{\geq 2} \times (\overline{\mathbb{Q}}^\star \cap \mathcal{D})\big)/\sim$. Then the canonical homomorphism of $\overline{\mathbb{Q}}$-algebras*

$$\bigotimes_{(q,\alpha)\in\mathcal{R}} \mathbf{M}_{\overline{\mathbb{Q}},q,\alpha} \longrightarrow \mathbf{M}, \qquad \bigotimes_{(q,\alpha)\in\mathcal{R}} x_{q,\alpha} \longmapsto \prod_{(q,\alpha)\in\mathcal{R}} x_{q,\alpha}$$

is an isomorphism. Here, the tensor product is taken over $\overline{\mathbb{Q}}$*, with* $x_{q,\alpha} = 1$ *for all but finitely many* (q, α) *in every elementary tensor.*

The study of linear and algebraic relations among the elements of $\mathbf{M}$ thus reduces to the corresponding problem within each of the smaller algebras $\mathbf{M}_{\overline{\mathbb{Q}},q,\alpha}$. Within each of these algebras, the lifting theorems [3, Th. 1.4] and [4, Th. 1] provide powerful tools for determining algebraic relations. This does not mean, however, that the remaining problem is always easy. Indeed, each algebra $\mathbf{M}_{\overline{\mathbb{Q}},q,\alpha}$ has infinite transcendence degree over $\overline{\mathbb{Q}}$. This follows, for instance, from Mahler's work [20], by considering the values at α of the Fredholm series[1]

$$\mathfrak{f}_q(z) = \sum_{n\geq 0} z^{q^n} \tag{1.2}$$

and of its derivatives; see also Section 1.2.

Theorem 1.1 also implies a linear disjointness result for M-functions, generalizing [5, Thm. 2.4] and [2, Thm. 1.8][2].

Corollary 1.4. *Let* $\mathcal{M}$ *be the* $\overline{\mathbb{Q}}(z)$*-algebra generated by all* M*-functions and, for every* $q \in \mathbb{N}_{\geq 2}$*, let* $\mathcal{M}_q$ *be the* $\overline{\mathbb{Q}}(z)$*-algebra generated by all* M_q*-functions. Set* $\mathcal{B} = \{q \geq 2 : q \text{ is not a perfect power}\}$*. Then the canonical homomorphism of* $\overline{\mathbb{Q}}(z)$*-algebras*

$$\bigotimes_{q\in\mathcal{B}} \mathcal{M}_q \longrightarrow \mathcal{M}, \qquad \bigotimes_{q\in\mathcal{B}} f_q \longmapsto \prod_{q\in\mathcal{B}} f_q$$

is an isomorphism. Here, the tensor product is taken over $\overline{\mathbb{Q}}(z)$*, with* $f_q = 1$ *for all but finitely many* q *in every elementary tensor.*

Since the generating series of q-automatic sequences are M_q-functions, this corollary generalizes Cobham's celebrated theorem on automatic sequences (see, for instance, the discussion in [5, Sec. 2.3]). In fact, it yields a substantially stronger statement: all linear and algebraic relations over $\overline{\mathbb{Q}}(z)$ among generating series of automatic sequences are obtained simply by combining the corresponding relations within each family of q-automatic generating series, where q ranges over the integers that are not perfect powers.

We also mention another linear disjointness result concerning base field extensions, recently obtained by the authors in [4, Cor. 3]. If $\mathbb{K} \subset \mathbb{L} \subset \overline{\mathbb{Q}}$, then

$$\mathbf{M}_{\mathbb{K},q,\alpha} \otimes_{\mathbb{K}} \mathbb{L} \simeq \mathbf{M}_{\mathbb{L},q,\alpha}.$$

This descent theorem further reduces the study of algebraic relations over $\overline{\mathbb{Q}}$ to the corresponding problem over the smaller field generated by the coefficients of the underlying power series and the evaluation point.

[1]The terminology was introduced by Kempner [12], following a paper of Mittag-Leffler that referred to work of Fredholm. The latter work, however, had no connection with the series under consideration, which would therefore more appropriately be called the Kempner series (see the discussions in [1, 31, 32]). Nevertheless, the name *Fredholm series* has become so widespread that we shall refrain from changing it.

[2]Theorem 1.8 in [2] is stated over the base field $\mathbb{C}$, but the extension from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ follows from a routine argument.

Theorem 1.1 is obtained as a consequence of two recent results of the authors on multivariate Mahler systems: a purity theorem and an extension of Nishioka's theorem; see [5, Cor. 3.5 and Cor. 3.9].

1.2. **Application to Fredholm series.** As an application of our general results, we give a complete description of all algebraic relations among the values of the Fredholm series introduced in (1.2) and their derivatives at nonzero algebraic points in the open unit disk. These series are among the oldest and most classical examples in Mahler's method. They appeared in Mahler's foundational work [19, 20, 21] and have since been repeatedly used as basic examples throughout the subject [13, 14, 15, 22, 25, 27, 23, 24]. Algebraic relations among their values have been explicitly studied in [13, 18, 26, 28, 29]. Nevertheless, a complete description involving arbitrary parameters, algebraic points, and derivatives was still missing.

There are several evident sources of relations among Fredholm series. They give rise to the following four rules:

(F) the *functional-equation rule* $\mathfrak{f}_q(\alpha) = \mathfrak{f}_q(\alpha^q) + \alpha$, which follows from the q-Mahler equation $\mathfrak{f}_q(z) = \mathfrak{f}_q(z^q) + z$;

(R) the *root-of-unity rule* $\mathfrak{f}_q(\zeta\alpha) = \sum_{i=0}^{\varphi(d)-1} \zeta^{q^i} \mathfrak{f}_{q^{\varphi(d)}}(\alpha^{q^i})$, where d is coprime to q, ζ is a primitive dth root of unity, and φ denotes Euler's totient function;

(E) the *exponentiation rule* $\mathfrak{f}_q(\alpha) = \sum_{i=0}^{n-1} \mathfrak{f}_{q^n}(\alpha^{q^i})$, $n \geq 2$, which relates values associated with the parameter q to values associated with the parameter q^n;

(D) the *derivative rule*, obtained by successively differentiating the functional identities underlying the rules (F), (R), and (E), and then specializing at algebraic points.

The exponentiation rule follows by splitting the defining series according to the residue class of the exponent index modulo n. Indeed, writing $k = nj + i$, with $0 \leq i \leq n-1$, we obtain

$$\mathfrak{f}_q(z) = \sum_{i=0}^{n-1} \sum_{j \geq 0} z^{q^{nj+i}} = \sum_{i=0}^{n-1} \mathfrak{f}_{q^n}(z^{q^i}). \tag{1.3}$$

Specialization at $z = \alpha$ gives rule (E). The root-of-unity rule is obtained by applying (1.3) with $z = \zeta\alpha$ and $n = \varphi(d)$, and using $\zeta^{q^{i+j\varphi(d)}} = \zeta^{q^i}$ for $j \geq 0$.

Our main result concerning the Fredholm series and their derivatives asserts that these evident relations generate all algebraic relations among the corresponding values. The difficulty is not to produce such relations, but to prove that no additional relation can arise from the interaction between distinct Mahler parameters, distinct algebraic points, and different orders of derivation. We state here an informal version; the precise formulation is given and proved in Section 5 (see Theorem 5.1).

Theorem 1.5. *Every algebraic relation among values of Fredholm series and their derivatives at nonzero algebraic points in the open unit disk can be obtained by successive applications of the rules* (F), (R), (E), *and* (D).

Thus, Theorem 1.5 provides an explicit description of the entire ideal of algebraic relations. This completely resolves a classical problem that has accompanied Mahler's method since its origins.

Remark 1.6. Let us illustrate Theorem 1.5 with an example. In [29], Nishioka concludes her paper with the following identity:

$$\sum_{k\geq 0}\alpha^{3^k}-\sqrt{-1}\sum_{k\geq 0}\bigl(\sqrt{-1}\,\alpha\bigr)^{3^k}-2\sum_{k\geq 0}\alpha^{9^k}=0. \tag{1.4}$$

She observes that this relation illustrates the difficulty of finding necessary and sufficient conditions for the algebraic independence of values of Fredholm series. In our notation, relation (1.4) reads

$$\mathfrak{f}_3(\alpha)-\sqrt{-1}\,\mathfrak{f}_3\bigl(\sqrt{-1}\,\alpha\bigr)-2\mathfrak{f}_9(\alpha)=0. \tag{1.5}$$

Applying rule (E) with $q=3$ and $n=2$ gives

$$\mathfrak{f}_3(\alpha)=\mathfrak{f}_9(\alpha)+\mathfrak{f}_9(\alpha^3). \tag{1.6}$$

On the other hand, applying rule (R) with $q=3$, $d=4$, and $\zeta=\sqrt{-1}$, we obtain

$$\mathfrak{f}_3\bigl(\sqrt{-1}\,\alpha\bigr)=\sqrt{-1}\,\mathfrak{f}_9(\alpha)-\sqrt{-1}\,\mathfrak{f}_9(\alpha^3),$$

since $\varphi(4)=2$, $\bigl(\sqrt{-1}\bigr)^3=-\sqrt{-1}$, and $3^{\varphi(4)}=9$. Multiplying the latter identity by $\sqrt{-1}$ and subtracting it from (1.6), we obtain

$$\mathfrak{f}_3(\alpha)-\sqrt{-1}\,\mathfrak{f}_3\bigl(\sqrt{-1}\,\alpha\bigr)=2\mathfrak{f}_9(\alpha),$$

which is equivalent to (1.5), and hence to (1.4). Thus, Nishioka's relation (1.4) is obtained by combining the elementary rules (E) and (R).

2. Background in Mahler's method

At the end of the 1920s, Mahler [19, 20, 21] developed a pioneering method for studying algebraic relations among values of M-functions. Important progress has recently made it possible to consider simultaneously values of M_q-functions associated with distinct parameters q and evaluated at several distinct algebraic points; see [5, 6]. We recall below the results that will be needed in the sequel.

Let n be a positive integer and let $T=(t_{i,j})_{1\leq i,j\leq n}$ be an $n\times n$ matrix with nonnegative integer entries. Given an n-tuple of indeterminates $\boldsymbol{z}=(z_1,\dots,z_n)$, we set

$$T\boldsymbol{z}:=\left(\prod_{j=1}^{n}z_j^{t_{1,j}},\dots,\prod_{j=1}^{n}z_j^{t_{n,j}}\right),$$

and we let T act similarly on $(\mathbb{C}^\star)^n$.

A *linear T-Mahler system*, or simply a *Mahler system*, is a system of functional equations of the form

$$\begin{pmatrix}f_1(\boldsymbol{z})\\ \vdots\\ f_m(\boldsymbol{z})\end{pmatrix}=A(\boldsymbol{z})\begin{pmatrix}f_1(T\boldsymbol{z})\\ \vdots\\ f_m(T\boldsymbol{z})\end{pmatrix}, \tag{2.1}$$

where $f_1(\boldsymbol{z}),\dots,f_m(\boldsymbol{z})\in\overline{\mathbb{Q}}[[\boldsymbol{z}]]$ and $A(\boldsymbol{z})\in\mathrm{GL}_m(\overline{\mathbb{Q}}(\boldsymbol{z}))$.

An algebraic point $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star)^n$ is said to be *regular with respect to* the system (2.1) if $A(\boldsymbol{z})$ is well defined and invertible at $T^\ell \boldsymbol{\alpha}$ for every $\ell \in \mathbb{N}$. In several variables, the application of Mahler's method requires an additional admissibility condition on the pair $(T, \boldsymbol{\alpha})$; see [5, Sec. 5].

In the present paper, however, we shall only need scalar matrices, that is, matrices of the form $T = q\mathrm{I}_n$ for some integer $q \geq 2$. In this case, $T\boldsymbol{z} = (z_1^q, \dots, z_n^q)$ and $T(\alpha_1, \dots, \alpha_n) = (\alpha_1^q, \dots, \alpha_n^q)$. This restriction allows one to express the admissibility condition on the pair $(T, \boldsymbol{\alpha})$ entirely in terms of the point $\boldsymbol{\alpha}$. Recall that $\mathcal{D}$ denotes the open unit disk. More precisely, the pair $(T = q\mathrm{I}_n, \boldsymbol{\alpha})$ is admissible in the sense of [5] if and only if $\boldsymbol{\alpha} \in (\overline{\mathbb{Q}}^\star \cap \mathcal{D})^n$ and the coordinates of $\boldsymbol{\alpha}$ are globally multiplicatively independent, that is,

$$\alpha_1^{u_1} \cdots \alpha_n^{u_n} = 1, \qquad (u_1, \dots, u_n) \in \mathbb{Z}^n,$$

implies $u_1 = \cdots = u_n = 0$.

The following result is a particular case of [5, Cor. 3.5]. It is a multivariate extension of Nishioka's theorem [27, Thm. 4.2.1].

Theorem 2.1. *Let n be a positive integer, $T = q\mathrm{I}_n$, and $f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z}) \in \overline{\mathbb{Q}}[[\boldsymbol{z}]]$ be related by a T-Mahler system. Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in (\overline{\mathbb{Q}}^\star \cap \mathcal{D})^n$. Assume that the coordinates of $\boldsymbol{\alpha}$ are globally multiplicatively independent and that $\boldsymbol{\alpha}$ is regular with respect to this system. Then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}\big(f_1(\boldsymbol{\alpha}), \dots, f_m(\boldsymbol{\alpha})\big) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}\big(f_1(\boldsymbol{z}), \dots, f_m(\boldsymbol{z})\big).$$

We shall also need a purity theorem for systems associated with multiplicatively independent parameters. Let r be a positive integer. For every $i \in \{1, \dots, r\}$, consider a T_i-Mahler system

$$\begin{pmatrix} f_{i,1}(\boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(\boldsymbol{z}_i) \end{pmatrix} = A_i(\boldsymbol{z}_i) \begin{pmatrix} f_{i,1}(T_i\boldsymbol{z}_i) \\ \vdots \\ f_{i,m_i}(T_i\boldsymbol{z}_i) \end{pmatrix}, \tag{2.1.i}$$

where n_i and m_i are positive integers, $\boldsymbol{z}_i = (z_{i,1}, \dots, z_{i,n_i})$ is a vector of indeterminates, $T_i = q_i\mathrm{I}_{n_i}$ with $q_i \geq 2$, $A_i(\boldsymbol{z}_i) \in \mathrm{GL}_{m_i}(\overline{\mathbb{Q}}(\boldsymbol{z}_i))$, and $f_{i,1}(\boldsymbol{z}_i), \dots, f_{i,m_i}(\boldsymbol{z}_i) \in \overline{\mathbb{Q}}[[\boldsymbol{z}_i]]$. Let also $\boldsymbol{\alpha}_i \in (\overline{\mathbb{Q}}^\star \cap \mathcal{D})^{n_i}$.

The following purity theorem is a particular case of [5, Cor. 3.9].

Theorem 2.2. *With the above notation, assume that the following conditions hold.*

(i) *For every $i \in \{1, \dots, r\}$, the coordinates of $\boldsymbol{\alpha}_i$ are globally multiplicatively independent, and $\boldsymbol{\alpha}_i$ is regular with respect to the corresponding T_i-Mahler system.*

(ii) *The integers $q_1, \dots, q_r$ are pairwise multiplicatively independent.*

For every $i \in \{1, \dots, r\}$, set $\mathcal{E}_i = \{f_{i,1}(\boldsymbol{\alpha}_i), \dots, f_{i,m_i}(\boldsymbol{\alpha}_i)\}$, and set $\mathcal{E} = \bigcup_{i=1}^r \mathcal{E}_i$. Then

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = \sum_{i=1}^r \mathrm{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_i).$$

3. Proof of Theorem 1.1

In this section, we prove our main general result concerning values of M-functions.

3.1. **Proof of part** (a) **of Theorem 1.1.** We first establish the following elementary stability lemma.

Lemma 3.1. *Let $n, q \geq 2$ be integers and let $f \in \overline{\mathbb{Q}}[[z]]$. Then the following assertions hold.*

(i) *If $f(z^n)$ is an M_q-function, then $f(z)$ is an M_q-function.*
(ii) *If $f(z)$ is an M_q-function, then $f(z^n)$ is an M_q-function.*
(iii) *Suppose that $f(z)$ is an M_q-function and write $f(z) = \sum_{i=0}^{n-1} z^i f_i(z^n)$, with $f_i(z) \in \overline{\mathbb{Q}}[[z]]$. Then each $f_i(z)$ is an M_q-function.*

Proof. Let us first prove (i). Consider a q-Mahler equation satisfied by $f(z^n)$:

$$a_0(z)f(z^n) + a_1(z)f(z^{qn}) + \cdots + a_m(z)f(z^{q^m n}) = 0, \tag{3.1}$$

where $a_0, \ldots, a_m \in \overline{\mathbb{Q}}[z]$ are not all zero. For every $r \in \{0, \ldots, m\}$, write $a_r(z) = \sum_{j=0}^{n-1} z^j a_{r,j}(z^n)$, with $a_{r,j}(z) \in \overline{\mathbb{Q}}[z]$. Substituting these decompositions into (3.1) and identifying the powers of z modulo n, we obtain, for every $j \in \{0, \ldots, n-1\}$,

$$a_{0,j}(z^n)f(z^n) + a_{1,j}(z^n)f(z^{qn}) + \cdots + a_{m,j}(z^n)f(z^{q^m n}) = 0.$$

Replacing z^n by z, we infer that

$$a_{0,j}(z)f(z) + a_{1,j}(z)f(z^q) + \cdots + a_{m,j}(z)f(z^{q^m}) = 0.$$

Since some a_i is nonzero, there exists a pair (i, j) such that $a_{i,j} \neq 0$. Hence f satisfies a nontrivial q-Mahler equation and is therefore an M_q-function. This proves (i).

Let us now prove (ii). Let

$$a_0(z)f(z) + a_1(z)f(z^q) + \cdots + a_m(z)f(z^{q^m}) = 0$$

be a nontrivial q-Mahler equation satisfied by f. Replacing z by z^n, we obtain

$$a_0(z^n)f(z^n) + a_1(z^n)f(z^{qn}) + \cdots + a_m(z^n)f(z^{q^m n}) = 0.$$

Thus, $f(z^n)$ is an M_q-function, proving (ii).

It remains to prove (iii). Since f is an M_q-function, it occurs as a coordinate of a vector $F(z) \in \overline{\mathbb{Q}}[[z]]^m$ satisfying a q-Mahler system

$$F(z) = A(z)F(z^q), \quad A(z) \in \mathrm{GL}_m(\overline{\mathbb{Q}}(z)).$$

Write $F(z) = \sum_{i=0}^{n-1} z^i F_i(z^n)$, with $F_i(z) \in \overline{\mathbb{Q}}[[z]]^m$. Then

$$\sum_{i=0}^{n-1} z^i F_i(z^n) = A(z) \sum_{i=0}^{n-1} z^{qi} F_i(z^{qn}) = \sum_{i=0}^{n-1} z^{qi} A(z) F_i(z^{qn}).$$

Identifying the powers of z modulo n, we see that every coordinate of every vector $F_i(z^n)$ is a linear combination over $\overline{\mathbb{Q}}(z)$ of the coordinates of $F_0(z^{qn}), \ldots, F_{n-1}(z^{qn})$.

Let $G(z)$ be the vector obtained by concatenating $F_0(z^n), \ldots, F_{n-1}(z^n)$. It follows that there exists a matrix $B(z) \in \mathcal{M}_{mn}(\overline{\mathbb{Q}}(z))$ such that $G(z) = B(z)G(z^q)$. By scalar elimination, every coordinate of a vector satisfying such a linear Mahler system satisfies a scalar q-Mahler equation; the matrix $B(z)$ need not be invertible for this conclusion. Thus, each $f_i(z^n)$ is an M_q-function. Assertion (i) then implies that each $f_i(z)$ is an M_q-function. □

Proof of part (a) *of Theorem 1.1.* Suppose first that $(q,\alpha) \not\sim (r,\beta)$. Then either q and r are multiplicatively independent, or α and β are multiplicatively independent. Let $\xi \in \mathbf{M}_{\mathbb{K},q,\alpha} \cap \mathbf{M}_{\mathbb{K},r,\beta}$. Then $\xi = f_1(\alpha) = f_2(\beta)$, where $f_1 \in \mathbb{K}[[z]]$ is an M_q-function and $f_2 \in \mathbb{K}[[z]]$ is an M_r-function. Since these two values are equal, they are algebraically dependent. It follows from [5, Thm. 1.1] that $\xi \in \mathbb{K}$. Hence $\mathbf{M}_{\mathbb{K},q,\alpha} \cap \mathbf{M}_{\mathbb{K},r,\beta} = \mathbb{K}$. Moreover, $\mathbf{M}_{\mathbb{K},q,\alpha} \neq \mathbb{K}$, since it contains the transcendental number $\mathfrak{f}_q(\alpha)$, where $\mathfrak{f}_q$ is the Fredholm series introduced in (1.2). Therefore, $\mathbf{M}_{\mathbb{K},q,\alpha} \neq \mathbf{M}_{\mathbb{K},r,\beta}$. This proves one implication.

Suppose now that $(q,\alpha) \sim (r,\beta)$. Then q and r are multiplicatively dependent, and so are α and β. Since $q, r \geq 2$, there exist positive integers u and v such that $q^u = r^v$.

We first note that, for every positive integer s, the classes of M_q-functions and M_{q^s}-functions coincide. Indeed, every M_q-function is an M_{q^s}-function, by iterating a q-Mahler equation. Conversely, every q^s-Mahler equation is also a q-Mahler equation. Consequently, $\mathbf{M}_{\mathbb{K},q,\alpha} = \mathbf{M}_{\mathbb{K},q^u,\alpha}$ and $\mathbf{M}_{\mathbb{K},r,\beta} = \mathbf{M}_{\mathbb{K},r^v,\beta}$. Replacing q and r by their common power $q^u = r^v$, which we again denote by q, we are reduced to proving that $\mathbf{M}_{\mathbb{K},q,\alpha} = \mathbf{M}_{\mathbb{K},q,\beta}$.

Since α and β are multiplicatively dependent and satisfy $0 < |\alpha|, |\beta| < 1$, there exist positive integers a and b such that $\alpha^a = \beta^b$. It is therefore enough to prove that, for every positive integer t, $\mathbf{M}_{\mathbb{K},q,\alpha} = \mathbf{M}_{\mathbb{K},q,\alpha^t}$. Indeed, applying this equality first to α and then to β, and using $\alpha^a = \beta^b$, we obtain $\mathbf{M}_{\mathbb{K},q,\alpha} = \mathbf{M}_{\mathbb{K},q,\alpha^a} = \mathbf{M}_{\mathbb{K},q,\beta^b} = \mathbf{M}_{\mathbb{K},q,\beta}$.

Let first $\xi \in \mathbf{M}_{\mathbb{K},q,\alpha^t}$. There exists an M_q-function $f \in \mathbb{K}[[z]]$ such that $\xi = f(\alpha^t)$. By Lemma 3.1(ii), the series $g(z) := f(z^t)$ is an M_q-function belonging to $\mathbb{K}[[z]]$. Since $\xi = g(\alpha)$, we obtain $\mathbf{M}_{\mathbb{K},q,\alpha^t} \subseteq \mathbf{M}_{\mathbb{K},q,\alpha}$.

Conversely, let $\xi \in \mathbf{M}_{\mathbb{K},q,\alpha}$. There exists an M_q-function $f \in \mathbb{K}[[z]]$ such that $\xi = f(\alpha)$. Write $f(z) = \sum_{i=0}^{t-1} z^i f_i(z^t)$, where $f_i(z) \in \mathbb{K}[[z]]$. By Lemma 3.1(iii), each f_i is an M_q-function. Hence $g(z) := \sum_{i=0}^{t-1} \alpha^i f_i(z)$ is also an M_q-function in $\mathbb{K}[[z]]$. By construction, $g(\alpha^t) = \sum_{i=0}^{t-1} \alpha^i f_i(\alpha^t) = f(\alpha) = \xi$. Thus, $\mathbf{M}_{\mathbb{K},q,\alpha} \subseteq \mathbf{M}_{\mathbb{K},q,\alpha^t}$. This proves $\mathbf{M}_{\mathbb{K},q,\alpha} = \mathbf{M}_{\mathbb{K},q,\alpha^t}$, which completes the proof. □

3.2. **Proof of part** (b) **of Theorem 1.1.** The proof is divided into three steps. First, for a fixed parameter q, we express values of M_q-functions at several algebraic points as values of functions satisfying a multivariate Mahler system at a regular and admissible point. Next, we use a functional purity theorem to separate the families of functions attached to pairwise multiplicatively independent points. Finally, we combine the multivariate extension of Nishioka's theorem with the purity theorem (Theorem 2.2) to establish the required algebraic independence statement.

Given $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{N}^n$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \overline{\mathbb{Q}}^n$, we use the standard monomial notation $\boldsymbol{\alpha}^{\boldsymbol{\mu}} := \alpha_1^{\mu_1} \cdots \alpha_n^{\mu_n}$. Similarly, if $\boldsymbol{z} = (z_1, \dots, z_n)$ is a vector of indeterminates, we set $\boldsymbol{z}^{\boldsymbol{\mu}} := z_1^{\mu_1} \cdots z_n^{\mu_n}$.

The following lemma is a straightforward generalization of [6, Prop. A.7]. In that proposition, only one function is considered at each point α_i. The same proof applies to finitely many functions at each point, by enlarging the corresponding Mahler systems.

Lemma 3.2. *Let $q \geq 2$ be an integer, and let $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}$ be pairwise multiplicatively independent algebraic numbers such that $0 < |\alpha_i| < 1$ for every i. For each $i \in \{1, \dots, r\}$, let $f_{i,1}(z), \dots, f_{i,s_i}(z) \in \overline{\mathbb{Q}}[[z]]$ be M_q-functions that are well defined at α_i.*

Then there exist positive integers t and ℓ, a point $\boldsymbol{\beta} \in \overline{\mathbb{Q}}^t$, a matrix $T \in \mathcal{M}_t(\mathbb{N})$, vectors $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r \in \mathbb{N}^t$, and, for each $i \in \{1, \dots, r\}$, a root of unity ζ_i, an integer $m_i \geq s_i$, and M_q-functions $g_{i,1}(z), \dots, g_{i,m_i}(z) \in \overline{\mathbb{Q}}[[z]]$, such that the following assertions hold.

(a) *For every $i \in \{1, \dots, r\}$, $\alpha_i = \zeta_i \boldsymbol{\beta}^{\boldsymbol{\mu}_i}$.*
(b) *For every $i \in \{1, \dots, r\}$ and $j \in \{1, \dots, s_i\}$, $f_{i,j}(\alpha_i) = g_{i,j}(\boldsymbol{\beta}^{\boldsymbol{\mu}_i})$.*
(c) *For every $i \in \{1, \dots, r\}$, the functions $g_{i,1}(z), \dots, g_{i,m_i}(z)$ are related by a q^ℓ-Mahler system, and $\boldsymbol{\beta}^{\boldsymbol{\mu}_i}$ is regular with respect to this system.*
(d) *The $m_1 + \cdots + m_r$ functions $g_{i,j}(\boldsymbol{z}^{\boldsymbol{\mu}_i})$, $1 \leq i \leq r$, $1 \leq j \leq m_i$, are related by a T-Mahler system, where $\boldsymbol{z} = (z_1, \dots, z_t)$ is a vector of indeterminates.*
(e) *The point $\boldsymbol{\beta}$ is regular for this T-Mahler system, its coordinates belong to $\overline{\mathbb{Q}}^\star \cap \mathcal{D}$, and they are globally multiplicatively independent.*
(f) *We have $T = q^\ell \mathrm{I}_t$.*
(g) *The vectors $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r$ are pairwise linearly independent over $\overline{\mathbb{Q}}$.*

The next result is [6, Lem. A.5].

Lemma 3.3. *Let r and t be positive integers, and let $\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r \in \mathbb{N}^t$ be pairwise linearly independent over $\overline{\mathbb{Q}}$. For each $i \in \{1, \dots, r\}$, let m_i be a positive integer and let $f_{i,1}(z), \dots, f_{i,m_i}(z) \in \overline{\mathbb{Q}}[[z]]$. If $\boldsymbol{z} = (z_1, \dots, z_t)$ is a vector of indeterminates, then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z})}\big(f_{i,j}(\boldsymbol{z}^{\boldsymbol{\mu}_i}) : 1 \leq i \leq r,\ 1 \leq j \leq m_i\big) = \sum_{i=1}^{r} \mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}\big(f_{i,j}(z) : 1 \leq j \leq m_i\big).$$

We now recall some standard facts concerning linear disjointness and freeness. The definitions given in the introduction extend the usual notions for field extensions. Indeed, two $\mathbb{K}$-algebras $\mathbb{E}$ and $\mathbb{F}$ are linearly disjoint over $\mathbb{K}$ if and only if their fields of fractions are linearly disjoint over $\mathbb{K}$. Likewise, they are free over $\mathbb{K}$ if and only if their fields of fractions are algebraically disjoint over $\mathbb{K}$, in the terminology of Bourbaki. We follow here the terminology of Lang; see [17, Chap. VIII] and [10, Chap. V, Sec. 1 and 14].

The corresponding statements also hold for arbitrary families of algebras. Moreover, if the field of fractions of each algebra in the family is a regular extension of $\mathbb{K}$, then linear disjointness and freeness are equivalent; see [17, Chap. VIII, Thm. 4.12] and [10, Chap. V, Sec. 14].

Recall that an integral $\mathbb{K}$-algebra $\mathbb{E}$ is said to be regular over $\mathbb{K}$ if its field of fractions $\mathrm{Frac}(\mathbb{E})$ is a regular extension of $\mathbb{K}$. In characteristic zero, this is equivalent to requiring that $\mathbb{K}$ be algebraically closed in $\mathrm{Frac}(\mathbb{E})$.

The next result is a rephrasing of [3, Cor. 1.8].

Lemma 3.4. *Let $\mathbb{K} \subset \overline{\mathbb{Q}}$. The algebras $\mathbf{M}_{\mathbb{K},q,\alpha}$, $q \in \mathbb{Z}_{\geq 2}$, $\alpha \in \mathbb{K}^\star \cap \mathcal{D}$, are regular over $\mathbb{K}$.*

We are now ready to prove part (b) of Theorem 1.1.

Proof of part (b) *of Theorem 1.1.* By Lemma 3.4, the field of fractions of each algebra $\mathbf{M}_{\mathbb{K},q,\alpha}$ is a regular extension of $\mathbb{K}$. It follows from [17, Chap. VIII, Thm. 4.12] that the algebras $\mathbf{M}_{\mathbb{K},q,\alpha}$, $(q,\alpha) \in \mathcal{I}$, are linearly disjoint over $\mathbb{K}$ if and only if they are free over $\mathbb{K}$.

Since freeness is a property of finite subfamilies, we may assume without loss of generality that $\mathcal{I}$ is finite. Let $(q_1,\alpha_1),\dots,(q_r,\alpha_r)$ be an enumeration of $\mathcal{I}$.

Using part (a) of Theorem 1.1, we may replace each pair (q_i,α_i) by an equivalent pair without changing the corresponding algebra. We may therefore assume that, for every $i<j$, either q_i and q_j are multiplicatively independent, or $q_i=q_j$ and α_i and α_j are multiplicatively independent.

For every $i \in \{1,\dots,r\}$, choose elements $\xi_{i,1},\dots,\xi_{i,s_i} \in \mathbf{M}_{\mathbb{K},q_i,\alpha_i}$ that are algebraically independent over $\mathbb{K}$. Since $\overline{\mathbb{Q}}/\mathbb{K}$ is algebraic, they are also algebraically independent over $\overline{\mathbb{Q}}$. We prove that the $s_1+\cdots+s_r$ numbers $\xi_{i,j}$, $1\le i\le r$, $1\le j\le s_i$, are algebraically independent over $\overline{\mathbb{Q}}$, and hence over $\mathbb{K}$. This will prove that the algebras $\mathbf{M}_{\mathbb{K},q_i,\alpha_i}$, $1\le i\le r$, are free over $\mathbb{K}$.

For every $i \in \{1,\dots,r\}$ and $j \in \{1,\dots,s_i\}$, choose an M_{q_i}-function $f_{i,j}(z) \in \mathbb{K}[[z]]$ such that $f_{i,j}(\alpha_i)=\xi_{i,j}$. Partition $\{1,\dots,r\}$ into classes $\mathcal{I}_1,\dots,\mathcal{I}_t$ so that i and j belong to the same class $\mathcal{I}_k$ if and only if $q_i = q_j =: \rho_k$.

For each $k\in\{1,\dots,t\}$, the points α_i, $i\in\mathcal{I}_k$, are pairwise multiplicatively independent. We may therefore apply Lemma 3.2 with $q=\rho_k$ to these points and to the corresponding functions $f_{i,1}(z),\dots,f_{i,s_i}(z)$. Let $\boldsymbol{\beta}_k$, $(\boldsymbol{\mu}_i)_{i\in\mathcal{I}_k}$, ℓ_k, T_k, $\boldsymbol{z}_k$, $(g_{i,j}(z))$ $(i\in\mathcal{I}_k,\ 1\le j\le m_i)$, and $B_k(\boldsymbol{z}_k)$ denote the point, vectors, iteration level, transformation matrix, vector of indeterminates, family of functions, and matrix of the resulting T_k-Mahler system.

Lemma 3.2 ensures that $\boldsymbol{\beta}_k$ is regular for the T_k-Mahler system defined by $B_k(\boldsymbol{z}_k)$, that its coordinates belong to $\overline{\mathbb{Q}}^\star \cap \mathcal{D}$, and that they are globally multiplicatively independent. Moreover, $T_k = \rho_k^{\ell_k}\mathrm{I}$. Since $\rho_1,\dots,\rho_t$ are pairwise multiplicatively independent, so are $\rho_1^{\ell_1},\dots,\rho_t^{\ell_t}$. We may therefore apply Theorem 2.2 to these t Mahler systems.

For every $k\in\{1,\dots,t\}$, set $\mathcal{F}_k := \{g_{i,j}(\boldsymbol{\beta}_k^{\boldsymbol{\mu}_i}) : i\in\mathcal{I}_k,\ 1\le j\le m_i\}$. Theorem 2.2 gives

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\left(\bigcup_{k=1}^{t}\mathcal{F}_k\right) = \sum_{k=1}^{t}\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_k).$$

Fix $k\in\{1,\dots,t\}$. Applying Theorem 2.1 to the T_k-Mahler system associated with $B_k(\boldsymbol{z}_k)$, we obtain

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_k) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z}_k)}\big(g_{i,j}(\boldsymbol{z}_k^{\boldsymbol{\mu}_i}) : i\in\mathcal{I}_k,\ 1\le j\le m_i\big).$$

Since the vectors $\boldsymbol{\mu}_i$, $i \in \mathcal{I}_k$, are pairwise linearly independent over $\overline{\mathbb{Q}}$, Lemma 3.3 yields

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}(\boldsymbol{z}_k)}\big(g_{i,j}(\boldsymbol{z}_k^{\boldsymbol{\mu}_i}) : i \in \mathcal{I}_k,\ 1 \le j \le m_i\big) = \sum_{i\in\mathcal{I}_k} \operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}\big(g_{i,j}(z) : 1 \le j \le m_i\big).$$

For every $i \in \mathcal{I}_k$, Condition (c) of Lemma 3.2 allows us to apply Theorem 2.1 to the Mahler system satisfied by $g_{i,1}(z), \dots, g_{i,m_i}(z)$ at the regular point $\boldsymbol{\beta}_k^{\boldsymbol{\mu}_i}$. Thus,

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}\big(g_{i,j}(z) : 1 \le j \le m_i\big) = \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_{k,i}),$$

where $\mathcal{F}_{k,i} := \{g_{i,j}(\boldsymbol{\beta}_k^{\boldsymbol{\mu}_i}) : 1 \le j \le m_i\}$. Let $\mathcal{F} = \bigcup_{k=1}^{t} \mathcal{F}_k$. Combining the preceding four identities, we obtain

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}) = \sum_{k=1}^{t} \sum_{i\in\mathcal{I}_k} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{F}_{k,i}). \tag{3.2}$$

For every $i \in \mathcal{I}_k$, set $\mathcal{E}_{k,i} := \{f_{i,1}(\alpha_i), \dots, f_{i,s_i}(\alpha_i)\}$ and

$$\mathcal{E} := \{f_{i,j}(\alpha_i) : 1 \le i \le r,\ 1 \le j \le s_i\}$$

By Condition (b) of Lemma 3.2, we have $\mathcal{E}_{k,i} \subseteq \mathcal{F}_{k,i}$. Moreover,

$$\mathcal{F} = \bigcup_{k=1}^{t} \bigcup_{i\in\mathcal{I}_k} \mathcal{F}_{k,i}, \quad \text{and} \quad \mathcal{E} = \bigcup_{k=1}^{t} \bigcup_{i\in\mathcal{I}_k} \mathcal{E}_{k,i}.$$

It follows from [5, Lem. 10.3] and (3.2) that

$$\begin{aligned} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) &= \sum_{k=1}^{t} \sum_{i\in\mathcal{I}_k} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}_{k,i}) \\ &= \sum_{i=1}^{r} \operatorname{tr.deg}_{\overline{\mathbb{Q}}}\big(f_{i,1}(\alpha_i), \dots, f_{i,s_i}(\alpha_i)\big). \end{aligned}$$

For every i, the numbers $f_{i,1}(\alpha_i), \dots, f_{i,s_i}(\alpha_i)$ are algebraically independent over $\overline{\mathbb{Q}}$, and hence $\operatorname{tr.deg}_{\overline{\mathbb{Q}}}\big(f_{i,1}(\alpha_i), \dots, f_{i,s_i}(\alpha_i)\big) = s_i$. Thus,

$$\operatorname{tr.deg}_{\overline{\mathbb{Q}}}(\mathcal{E}) = s_1 + \cdots + s_r.$$

It follows that all the numbers $\xi_{i,j}$, $1 \le i \le r$, $1 \le j \le s_i$, are algebraically independent over $\overline{\mathbb{Q}}$, and therefore over $\mathbb{K}$. Hence the algebras $\mathbf{M}_{\mathbb{K},q_i,\alpha_i}$, $1 \le i \le r$, are free over $\mathbb{K}$, and thus linearly disjoint over $\mathbb{K}$. This ends the proof. □

4. Proof of Corollary 1.4

Corollary 1.4 is equivalent to saying that the $\overline{\mathbb{Q}}(z)$-algebras $\mathcal{M}_q$, where $q \in \mathcal{B}$, are linearly disjoint over $\overline{\mathbb{Q}}(z)$. By [3, Lem. 3.2], each algebra $\mathcal{M}_q$ is a regular extension of $\overline{\mathbb{Q}}(z)$. It follows that the algebras $\mathcal{M}_q$, $q \in \mathcal{B}$, are linearly disjoint over $\overline{\mathbb{Q}}(z)$ if and only if they are free over $\overline{\mathbb{Q}}(z)$.

Let $q_1, \dots, q_r \in \mathcal{B}$ be pairwise distinct and, for every $i \in \{1, \dots, r\}$, let $f_{i,1}, \dots, f_{i,s_i} \in \mathcal{M}_{q_i}$ be algebraically independent over $\overline{\mathbb{Q}}(z)$. We only have to

prove that the $s_1+\cdots+s_r$ series $f_{i,j}$, $1\le i\le r$, $1\le j\le s_i$, are algebraically independent over $\overline{\mathbb{Q}}(z)$.

Let $R<1$ be a positive real number. By Case (M_q) of [4, Prop. 14], for each i, there exists a finite set $\mathcal{S}_i$ such that the following holds: for any $\alpha\in\overline{\mathbb{Q}}^\star\setminus\mathcal{S}_i$ with $|\alpha|<R$, the numbers $f_{i,1}(\alpha),\dots,f_{i,s_i}(\alpha)$ are algebraically independent.

Since each set $\mathcal{S}_i$ is finite, there exists $\alpha\in\overline{\mathbb{Q}}^\star\setminus\bigcup_i\mathcal{S}_i$ with $|\alpha|<R$. For such an α and for every i, the s_i numbers $f_{i,1}(\alpha),\dots,f_{i,s_i}(\alpha)$ are algebraically independent. These numbers belong to the algebra $\mathbf{M}_{\overline{\mathbb{Q}},q_i,\alpha}$. Since the $q_1,\dots,q_r\in\mathcal{B}$ are pairwise distinct, the pairs (q_i,α), $i\in\{1,\dots,r\}$ are pairwise inequivalent with respect to $\sim$. It follows from part (b) of Theorem 1.1, applied with $\mathbb{K}=\overline{\mathbb{Q}}$ and $\mathcal{I}=\{(q_1,\alpha),\dots,(q_r,\alpha)\}$, that the algebras $\mathbf{M}_{\overline{\mathbb{Q}},q_1,\alpha},\dots,\mathbf{M}_{\overline{\mathbb{Q}},q_r,\alpha}$ are linearly disjoint over $\overline{\mathbb{Q}}$. In particular, the $s_1+\cdots+s_r$ numbers

$$f_{i,j}(\alpha),\quad 1\le i\le r,\ 1\le j\le s_i$$

are algebraically independent over $\overline{\mathbb{Q}}$. A fortiori, the $s_1+\cdots+s_r$ series $f_{i,j}$, $1\le i\le r$, $1\le j\le s_i$, are algebraically independent over $\overline{\mathbb{Q}}(z)$. □

5. Algebraic relations among values of Fredholm series

The goal of this section is to give a precise formulation of Theorem 1.5 and to prove it. To simplify the exposition, we use the logarithmic derivation, or Euler operator, $\theta:=z\frac{\partial}{\partial z}$, instead of the usual derivation. For every nonnegative integer ℓ, we set

$$\mathfrak{f}_q^{(\ell)}(z):=\theta^\ell\big(\mathfrak{f}_q\big)(z)=\sum_{n=0}^{\infty}q^{\ell n}z^{q^n}.$$

We introduce the $\overline{\mathbb{Q}}$-algebra generated by all values of Fredholm series and their successive Euler derivatives at nonzero algebraic points in the open unit disk:

$$\mathcal{F}:=\overline{\mathbb{Q}}\left[\mathfrak{f}_q^{(\ell)}(\alpha):q\in\mathbb{Z}_{\ge2},\ \ell\in\mathbb{Z}_{\ge0},\ \alpha\in\overline{\mathbb{Q}}^\star\cap\mathcal{D}\right]\subseteq\mathbb{C}.$$

We also introduce the polynomial ring over $\overline{\mathbb{Q}}$ in the corresponding countable family of indeterminates:

$$\mathcal{A}:=\overline{\mathbb{Q}}\left[X_{q,\ell,\alpha}:q\in\mathbb{Z}_{\ge2},\ \ell\in\mathbb{Z}_{\ge0},\ \alpha\in\overline{\mathbb{Q}}^\star\cap\mathcal{D}\right].$$

We define the evaluation map

$$\begin{aligned}\mathrm{ev}:\mathcal{A}&\longrightarrow\mathcal{F},\\ X_{q,\ell,\alpha}&\longmapsto\mathfrak{f}_q^{(\ell)}(\alpha).\end{aligned}$$

By definition, ev is a surjective homomorphism of $\overline{\mathbb{Q}}$-algebras.

Let $\boldsymbol{X} = (X_{q,\ell,\alpha})_{q,\ell,\alpha}$ denote the corresponding family of indeterminates, so that $\mathcal{A} = \overline{\mathbb{Q}}[\boldsymbol{X}]$. We consider the following polynomials:

$$\begin{aligned} F_{q,\ell,\alpha}(\boldsymbol{X}) &= X_{q,\ell,\alpha} - q^\ell X_{q,\ell,\alpha^q} - \alpha, \\ R_{\zeta,q,\ell,\alpha}(\boldsymbol{X}) &= X_{q,\ell,\zeta\alpha} - \sum_{i=0}^{\varphi(d)-1} \zeta^{q^i} q^{i\ell} X_{q^{\varphi(d)},\ell,\alpha^{q^i}}, \\ E_{n,q,\ell,\alpha}(\boldsymbol{X}) &= X_{q,\ell,\alpha} - \sum_{i=0}^{n-1} q^{i\ell} X_{q^n,\ell,\alpha^{q^i}}. \end{aligned} \tag{5.1}$$

Here, $q, n, d \geq 2$, $\ell \in \mathbb{Z}_{\geq 0}$, $\alpha \in \overline{\mathbb{Q}}^\star \cap \mathcal{D}$, the integers d and q are coprime, and ζ is a primitive dth root of unity.

When $\ell = 0$, these polynomials correspond respectively to the relations arising from the rules (F), (R), and (E) introduced in Section 1.2. For $\ell \geq 1$, they correspond to the relations obtained by applying θ^ℓ to the underlying functional identities and then specializing at α, in accordance with rule (D).

We let $\mathcal{I}$ denote the ideal of $\mathcal{A}$ generated by all the polynomials in (5.1). Thus, $\mathcal{I}$ is the ideal generated by the relations obtained from the rules (F), (R), (E), and (D).

Theorem 1.5 can now be stated precisely as follows.

Theorem 5.1. *We have* $\ker(\mathrm{ev}) = \mathcal{I}$. *Equivalently, the evaluation map induces an isomorphism of* $\overline{\mathbb{Q}}$*-algebras*

$$\overline{\mathrm{ev}} : \mathcal{A}/\mathcal{I} \xrightarrow{\sim} \mathcal{F}.$$

The proof of Theorem 5.1 is divided into four steps. First, we show that $\mathcal{I} \subseteq \ker(\mathrm{ev})$. To establish the reverse inclusion, we first apply classical results from Mahler's method in one variable to exhibit families of values that are algebraically independent. Using the polynomials in (5.1), we deduce that any relation involving only pairs (q, α) that are pairwise equivalent belongs to $\mathcal{I}$. Then, we leverage the linear disjointness of the algebras $\mathbf{M}_{\overline{\mathbb{Q}},q,\alpha}$ to show that the latter relations generate the entire ideal $\ker(\mathrm{ev})$. We conclude that $\ker(\mathrm{ev}) = \mathcal{I}$.

5.1. **Elements of $\mathcal{I}$ are relations.** We begin by showing that the polynomials in (5.1) belong to the kernel of ev.

Lemma 5.2. *We have* $\mathcal{I} \subseteq \ker(\mathrm{ev})$.

We recall the following two elementary identities:

$$\theta^\ell(f(\xi z)) = \theta^\ell(f)(\xi z) \quad \text{and} \quad \theta^\ell(f(z^r)) = r^\ell \theta^\ell(f)(z^r),$$

where f is a power series, r is a positive integer, and ξ is a complex number.

Proof. We first consider the polynomials $F_{q,\ell,\alpha}$. Applying θ^ℓ to the q-Mahler equation $\mathfrak{f}_q(z) = \mathfrak{f}_q(z^q) + z$ and then specializing at α, we obtain

$$\mathrm{ev}\big(F_{q,\ell,\alpha}\big) = \mathfrak{f}_q^{(\ell)}(\alpha) - q^\ell \mathfrak{f}_q^{(\ell)}(\alpha^q) - \alpha = 0.$$

Thus, $F_{q,\ell,\alpha} \in \ker(\mathrm{ev})$.

We next consider the polynomials $R_{\zeta,q,\ell,\alpha}$, where ζ is a primitive dth root of unity with d and q relatively prime. Applying θ^ℓ to the identity

$$\mathfrak{f}_q(\zeta z) = \sum_{i=0}^{\varphi(d)-1} \zeta^{q^i} \mathfrak{f}_{q^{\varphi(d)}}(z^{q^i})$$

and then specializing at α, we obtain

$$\mathrm{ev}\big(R_{\zeta,q,\ell,\alpha}\big) = \mathfrak{f}_q^{(\ell)}(\zeta\alpha) - \sum_{i=0}^{\varphi(d)-1} \zeta^{q^i} q^{i\ell} \mathfrak{f}_{q^{\varphi(d)}}^{(\ell)}(\alpha^{q^i}) = 0.$$

Therefore, $R_{\zeta,q,\ell,\alpha} \in \ker(\mathrm{ev})$.

Finally, we consider the polynomials $E_{n,q,\ell,\alpha}$. Applying θ^ℓ to the identity $\mathfrak{f}_q(z) = \sum_{i=0}^{n-1} \mathfrak{f}_{q^n}(z^{q^i})$ and then specializing at α, we obtain

$$\mathrm{ev}\big(E_{n,q,\ell,\alpha}\big) = \mathfrak{f}_q^{(\ell)}(\alpha) - \sum_{i=0}^{n-1} q^{i\ell} \mathfrak{f}_{q^n}^{(\ell)}(\alpha^{q^i}) = 0.$$

Hence, $E_{n,q,\ell,\alpha} \in \ker(\mathrm{ev})$.

Since $\ker(\mathrm{ev})$ is an ideal of $\mathcal{A}$, it contains the ideal generated by all the polynomials in (5.1). Consequently, $\mathcal{I} \subseteq \ker(\mathrm{ev})$. □

5.2. **Algebraic independence of subfamilies of values.** Fix an integer $q \geq 2$ and an algebraic number $\alpha \in \overline{\mathbb{Q}}^\star \cap \mathcal{D}$. Define an equivalence relation over $\mathbb{Z}_{>0}$, setting

$$b \sim_q c \quad \Longleftrightarrow \quad \frac{b}{c} = q^a \quad \text{for some } a \in \mathbb{Z}. \tag{5.2}$$

Let $(c_\nu)_{\nu\geq 0} \subset \mathbb{Z}_{>0}$ denote an enumeration of a set of representatives of the equivalence classes with respect to $\sim_q$.

Lemma 5.3. *The numbers $\mathfrak{f}_q^{(\ell)}(\alpha^{c_\nu})$, $\ell \geq 0$, $\nu \geq 0$, are algebraically independent.*

Proof. It is sufficient to prove that, for any integers $s, t \geq 0$, the st numbers

$$\mathfrak{f}_q^{(\ell)}(\alpha^{c_\nu}), \quad 0 \leq \ell < s,\ 0 \leq \nu < t,$$

are algebraically independent. Set

$$g_{\ell,\nu}(z) := \mathfrak{f}_q^{(\ell)}(z^{c_\nu}), \quad 0 \leq \ell < s,\ 0 \leq \nu < t.$$

They satisfy the following q-Mahler equation:

$$g_{\ell,\nu}(z) = q^\ell g_{\ell,\nu}(z^q) + z^{c_\nu}.$$

Hence the function 1 together with the st functions $g_{\ell,\nu}(z)$, $0 \leq \ell < s$, $0 \leq \nu < t$, satisfies a q-Mahler system of size $st+1$ whose matrix has polynomial entries and nonzero constant determinant. In particular, α is regular with respect to this system.

We claim that the functions $g_{\ell,\nu}(z)$, $0 \leq \ell < s$, $0 \leq \nu < t$, are algebraically independent over $\overline{\mathbb{Q}}(z)$. Otherwise, [21, Satz 1], or equivalently [15, Cor. 9], would imply the existence of algebraic numbers $\mu_{\ell,\nu}$, not all zero, such that

$$\sum_{\ell=0}^{s-1} \sum_{\nu=0}^{t-1} \mu_{\ell,\nu} g_{\ell,\nu}(z) \in \overline{\mathbb{Q}}(z).$$

This power series is equal to

$$\sum_{\nu=0}^{t-1}\sum_{n\geq 0}\left(\sum_{\ell=0}^{s-1}\mu_{\ell,\nu}q^{\ell n}\right)z^{c_\nu q^n}.$$

Since the integers c_ν belong to distinct equivalence classes with respect to $\sim_q$, the sets $\{c_\nu q^n : n\geq 0\}$, $0\leq \nu < t$, are pairwise disjoint. Their union has arbitrarily long gaps. A rational power series supported on such a set must therefore be a polynomial. It follows that, for every ν,

$$\sum_{\ell=0}^{s-1}\mu_{\ell,\nu}q^{\ell n}=0$$

for all sufficiently large n. A Vandermonde argument gives $\mu_{\ell,\nu}=0$ for every $\ell\in\{0,\dots,s-1\}$ and every $\nu\in\{0,\dots,t-1\}$, a contradiction.

Then, Nishioka's theorem, *i.e.* Theorem 2.1 with $n=1$, applied at α implies that

$$\begin{aligned}\operatorname{tr.deg}_{\overline{\mathbb{Q}}}((\mathfrak{f}_q^{(\ell)}(\alpha^{c_\nu}))_{0\leq\ell<s,\,0\leq\nu<t}) &= \operatorname{tr.deg}_{\overline{\mathbb{Q}}}(1,(g_{\ell,\nu}(\alpha))_{0\leq\ell<s,\,0\leq\nu<t})\\ &= \operatorname{tr.deg}_{\overline{\mathbb{Q}}(z)}(1,(g_{\ell,\nu}(z))_{0\leq\ell<s,\,0\leq\nu<t})\\ &= st\,.\end{aligned}$$

Therefore, the st numbers $\mathfrak{f}_q^{(\ell)}(\alpha^{c_\nu})$, $0\leq\ell<s$, $0\leq\nu<t$, are algebraically independent over $\overline{\mathbb{Q}}$. Note that the full strength of Nishioka's theorem is not needed here. Alternatively, one could use the main result of Mahler's paper [21], with $n=1$. □

5.3. **Relations within each equivalence class.** Recall that the equivalence relation $\sim$ on pairs $(q,\alpha)\in\mathbb{Z}_{\geq 2}\times(\overline{\mathbb{Q}}^\star\cap\mathcal{D})$ was defined by

$$(q,\alpha)\sim(p,\beta)\quad\Longleftrightarrow\quad q\sim p \text{ and } \alpha\sim\beta,$$

where, on the right-hand side, $\sim$ denotes multiplicative dependence of nonzero complex numbers.

Let $(\mathcal{C}_i)_{i\in\mathbb{N}}$ denote an enumeration of the equivalence classes for this relation. For every $i\in\mathbb{N}$, let $\boldsymbol{X}_i$ be the subfamily of indeterminates $X_{q,\ell,\alpha}$ such that $(q,\alpha)\in\mathcal{C}_i$.

Lemma 5.4. *For every $i\in\mathbb{N}$, we have* $\ker(\mathrm{ev})\cap\overline{\mathbb{Q}}[\boldsymbol{X}_i]\subset\mathcal{I}$.

Proof. Let $P_0\in\ker(\mathrm{ev})\cap\overline{\mathbb{Q}}[\boldsymbol{X}_i]$. There exist integers $q_1,\dots,q_s\geq 2$, nonnegative integers $\ell_1,\dots,\ell_s$, and algebraic numbers $\alpha_1,\dots,\alpha_s\in\overline{\mathbb{Q}}^\star\cap\mathcal{D}$ such that

$$P_0\in\overline{\mathbb{Q}}[X_{q_1,\ell_1,\alpha_1},\dots,X_{q_s,\ell_s,\alpha_s}].$$

Since all the pairs (q_k,α_k) belong to the same equivalence class $\mathcal{C}_i$, the integers $q_1,\dots,q_s$ are pairwise multiplicatively dependent, and the numbers $\alpha_1,\dots,\alpha_s$ are pairwise multiplicatively dependent.

There exist an integer $q\geq 2$ and positive integers $u_1,\dots,u_s$ such that

$$q_k=q^{u_k},\qquad 1\leq k\leq s.$$

There also exist an algebraic number $\alpha\in\overline{\mathbb{Q}}^\star\cap\mathcal{D}$, roots of unity $\zeta_1,\dots,\zeta_s$, and positive integers $v_1,\dots,v_s$ such that

$$\alpha_k=\zeta_k\alpha^{v_k},\qquad 1\leq k\leq s.$$

We first remove from each ζ_k the part of its order supported on the prime divisors of q. The relation

$$F_{q_k,\ell_k,\beta} = X_{q_k,\ell_k,\beta} - q_k^{\ell_k} X_{q_k,\ell_k,\beta^{q_k}} - \beta$$

shows that, modulo $\mathcal{I}$, the indeterminate $X_{q_k,\ell_k,\beta}$ is equal to

$$q_k^{\ell_k} X_{q_k,\ell_k,\beta^{q_k}} + \beta.$$

Applying this operation sufficiently many times, we obtain

$$P_0 = P_1 + Q_0, \qquad Q_0 \in \mathcal{I},$$

where P_1 involves only indeterminates of the form $X_{q_k,\ell_k,\eta_k\alpha^{a_k}}$, with $a_k \geq 1$ and η_k a root of unity whose order d_k is coprime with q, and hence with q_k.

The relation

$$R_{\eta_k,q_k,\ell_k,\alpha^{a_k}} = X_{q_k,\ell_k,\eta_k\alpha^{a_k}} - \sum_{h=0}^{\varphi(d_k)-1} \eta_k^{q_k^h} q_k^{h\ell_k} X_{q_k^{\varphi(d_k)},\ell_k,\alpha^{a_k q_k^h}}$$

then allows us to eliminate all roots of unity. Thus,

$$P_1 = P_2 + Q_1, \qquad Q_1 \in \mathcal{I},$$

where P_2 involves only indeterminates of the form $X_{r_k,\ell_k,\alpha^{a_k}}$, and the integers r_k are powers of q.

Choose a common power r of the integers r_k. Thus, for every k, there exists a positive integer n_k such that $r_k^{n_k} = r$. Using the relation

$$E_{n_k,r_k,\ell_k,\alpha^{a_k}} = X_{r_k,\ell_k,\alpha^{a_k}} - \sum_{h=0}^{n_k-1} r_k^{h\ell_k} X_{r,\ell_k,\alpha^{a_k r_k^h}},$$

we obtain

$$P_2 = P_3 + Q_2, \qquad Q_2 \in \mathcal{I},$$

where P_3 involves only indeterminates of the form X_{r,ℓ,α^b}, $\ell \geq 0$, $b \geq 1$.

Consider an enumeration $(c_\nu)_{\nu\geq 0}$ of a set of representatives for the equivalence relation $\sim_r$ introduced in (5.2) such that $r \nmid c_\nu$, for every ν. Then, each element in the equivalence class of c_ν is of the form $c_\nu r^a$ with $a \geq 0$. Set $\beta_\nu := \alpha^{c_\nu}$. Every indeterminate appearing in P_3 is of the form $X_{r,\ell,\alpha^{c_\nu r^a}} = X_{r,\ell,\beta_\nu^{r^a}}$, for some $\nu, a \geq 0$.

Repeated use of the relations

$$F_{r,\ell,\beta_\nu^{r^{a-1}}} = X_{r,\ell,\beta_\nu^{r^{a-1}}} - r^\ell X_{r,\ell,\beta_\nu^{r^a}} - \beta_\nu^{r^{a-1}}, \quad a \geq 1,$$

allows us to replace every indeterminate $X_{r,\ell,\beta_\nu^{r^a}}$ by an expression involving X_{r,ℓ,β_ν} and an algebraic constant. Thus, we may replace every indeterminate X_{r,ℓ,α^b} by an expression involving $X_{r,\ell,\alpha^{c_\nu}}$ for some $\nu \geq 0$. We therefore obtain

$$P_3 = P_4 + Q_3, \qquad Q_3 \in \mathcal{I},$$

where P_4 involves only indeterminates of the form $X_{r,\ell,\alpha^{c_\nu}}$, $\ell \geq 0$, $\nu \geq 0$.

We have the decomposition

$$P_0 = Q_0 + Q_1 + Q_2 + Q_3 + P_4.$$

By Lemma 5.2, $\mathcal{I} \subset \ker(\mathrm{ev})$, so Q_0, Q_1, Q_2, and Q_3 all belong to $\ker(\mathrm{ev})$. Since $P_0 \in \ker(\mathrm{ev})$ as well, we have $P_4 \in \ker(\mathrm{ev})$. The polynomial P_4 involves only indeterminates of the form $X_{r,\ell,\alpha^{c_\nu}}$ with $\ell \geq 0$ and $\nu \geq 0$.

Lemma 5.3 implies that the numbers $\mathfrak{f}_r^{(\ell)}(\alpha^{c_\nu})$ are algebraically independent. Thus, $P_4 = 0$ and

$$P_0 = Q_0 + Q_1 + Q_2 + Q_3 \in \mathcal{I}.$$

This proves the inclusion $\ker(\mathrm{ev}) \cap \overline{\mathbb{Q}}[\boldsymbol{X}_i] \subset \mathcal{I}$.

□

5.4. **Linear disjointness.** We continue with the notation $\mathcal{C}_i$ and $\boldsymbol{X}_i$ from the previous subsection.

Lemma 5.5. *The ideal* $\ker(\mathrm{ev})$ *is generated by the elements of the sets* $\ker(\mathrm{ev}) \cap \overline{\mathbb{Q}}[\boldsymbol{X}_i]$, $i \in \mathbb{N}$.

Proof. For every $i \in \mathbb{N}$, choose a pair $(q, \alpha) \in \mathcal{C}_i$ and set $\mathbf{M}_i := \mathbf{M}_{\overline{\mathbb{Q}},q,\alpha}$. By Theorem 1.1(a), this algebra does not depend on the choice of (q, α) in $\mathcal{C}_i$.

If $(q, \alpha) \in \mathcal{C}_i$ and $\ell \in \mathbb{Z}_{\geq 0}$, then

$$\mathrm{ev}(X_{q,\ell,\alpha}) = \mathfrak{f}_q^{(\ell)}(\alpha) \in \mathbf{M}_i,$$

since $\mathfrak{f}_q^{(\ell)}$ is an M_q-function. Hence the image of the restriction

$$\mathrm{ev}_i : \overline{\mathbb{Q}}[\boldsymbol{X}_i] \longrightarrow \mathbb{C}$$

is a subalgebra of $\mathbf{M}_i$. Let $\mathcal{K}$ be a finite subset of $\mathbb{N}$, and set

$$\mathcal{A}_\mathcal{K} := \overline{\mathbb{Q}}\left[\bigcup_{i\in\mathcal{K}} \boldsymbol{X}_i\right].$$

Since the families $\boldsymbol{X}_i$, $i \in \mathcal{K}$, are pairwise disjoint, there is a natural isomorphism

$$\mathcal{A}_\mathcal{K} \simeq \bigotimes_{i\in\mathcal{K}} \overline{\mathbb{Q}}[\boldsymbol{X}_i],$$

where the tensor products are taken over $\overline{\mathbb{Q}}$. For every $i \in \mathcal{K}$, set

$$\mathcal{J}_i := \ker(\mathrm{ev}) \cap \overline{\mathbb{Q}}[\boldsymbol{X}_i].$$

We obtain a natural surjective homomorphism

$$\mathcal{A}_\mathcal{K} \longrightarrow \bigotimes_{i\in\mathcal{K}} \overline{\mathbb{Q}}[\boldsymbol{X}_i]/\mathcal{J}_i,$$

whose kernel is the ideal of $\mathcal{A}_\mathcal{K}$ generated by the elements of $\mathcal{J}_i$, $i \in \mathcal{K}$. Since the classes $\mathcal{C}_i$, $i \in \mathcal{K}$, are distinct, Theorem 1.1 implies that the algebras $\mathbf{M}_i$, $i \in \mathcal{K}$, are linearly disjoint over $\overline{\mathbb{Q}}$. Consequently, their subalgebras $\mathrm{ev}_i\big(\overline{\mathbb{Q}}[\boldsymbol{X}_i]\big)$, $i \in \mathcal{K}$, are also linearly disjoint over $\overline{\mathbb{Q}}$. It follows that the multiplication map

$$\bigotimes_{i\in\mathcal{K}} \overline{\mathbb{Q}}[\boldsymbol{X}_i]/\mathcal{J}_i \longrightarrow \mathbb{C}$$

induced by evaluation is injective. Therefore, $\ker(\mathrm{ev}) \cap \mathcal{A}_\mathcal{K}$ is precisely the ideal of $\mathcal{A}_\mathcal{K}$ generated by the elements of $\mathcal{J}_i$, $i \in \mathcal{K}$.

Finally, every polynomial in $\mathcal{A}$ involves indeterminates from only finitely many families $\boldsymbol{X}_i$. Taking $\mathcal{K}$ to be the corresponding finite set of indices proves that $\ker(\mathrm{ev})$ is generated by the elements of the sets $\ker(\mathrm{ev}) \cap \overline{\mathbb{Q}}[\boldsymbol{X}_i]$, for $i \in \mathbb{N}$. □

5.5. **End of the proof of Theorem 5.1.** By Lemma 5.2, we already know that $\mathcal{I} \subseteq \ker(\mathrm{ev})$. It remains to prove the reverse inclusion.

Lemma 5.5 shows that $\ker(\mathrm{ev})$ is generated by the elements of the sets $\ker(\mathrm{ev}) \cap \overline{\mathbb{Q}}[\boldsymbol{X}_i]$, for $i \in \mathbb{N}$. Furthermore, by Lemma 5.4, $\ker(\mathrm{ev}) \cap \overline{\mathbb{Q}}[\boldsymbol{X}_i] \subset \mathcal{I}$ for every $i \in \mathbb{N}$. Therefore, $\ker(\mathrm{ev}) \subseteq \mathcal{I}$, which completes the proof of Theorem 5.1. □

Declaration on the use of AI. All new mathematical ideas and results presented in this article are entirely due to the authors. AI tools were used solely to check spelling and identify typographical errors in mathematical expressions.

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