THE SIEGEL–SHIDLOVSKII METHOD REVISITED

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Abstract. We revisit the classical Siegel–Shidlovskii method in transcendental number theory from a new perspective, in which functional relations play a central role. Within a single self-contained framework, we recover the foundational results in the theory of E-functions, including the Siegel–Shidlovskii theorem, Beukers' lifting theorem, and André's theorem on the singularities of the minimal differential operator annihilating an E-function. Notably, the latter is obtained without any recourse to the theory of G-functions. This unified approach also allows us to resolve a problem posed by Shidlovskii: we obtain a general algebraic independence measure extending the classical Siegel–Lang measure to an arbitrary algebraically independent family of E-values. We also provide a short elementary proof of a second theorem of André, which has become an important tool in the theory of E-functions and concerns the primality of the specialization, at a regular point, of the ideal of functional relations of a vector of E-functions satisfying a differential system.

1. Introduction

In his landmark memoir of 1929, Siegel [44] introduced a method for proving the transcendence and algebraic independence of values at algebraic points of certain power series solutions of linear differential equations, which he called E-functions because of their similarity with the exponential function. His aim was to extend the theory of Hermite, which culminates in the Lindemann–Weierstrass theorem, beyond the exponential function, a goal he achieved for the Bessel function.

A decisive breakthrough was later made by Shidlovskii, who removed the normality assumption introduced by Siegel in his 1949 monograph [45], thereby eliminating the main obstacle to the general application of the method. This led to what is now known as the Siegel–Shidlovskii Theorem: if $f_1, \dots, f_m$ are E-functions satisfying a differential system

$$Y'(z) = A(z)Y(z), \qquad A \in \mathcal{M}_m(\overline{\mathbb{Q}}(z)), \tag{1.1}$$

and if $\alpha \in \overline{\mathbb{Q}}^\times$ is regular with respect to (1.1), *i.e.* not a pole of $A(z)$, then

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}\big(f_1(\alpha), \dots, f_m(\alpha)\big) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}\big(f_1(z), \dots, f_m(z)\big).$$

In particular, the algebraic independence of $f_1, \dots, f_m$ over $\overline{\mathbb{Q}}(z)$ implies that of $f_1(\alpha), \dots, f_m(\alpha)$ over $\overline{\mathbb{Q}}$.

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The subsequent development of the theory followed, for the most part, three major directions.

(i) The functional algebraic independence problem underlying the application of the Siegel–Shidlovskii Theorem: developing methods to prove the algebraic independence of solutions of differential systems of the form (1.1) and applying them to concrete examples. This led to numerous criteria, developed in particular by Shidlovskii and his school (see [37] and the references therein), and later to methods based on differential Galois theory, culminating in the work of Beukers, Brownawell, and Heckman [11].

(ii) Refinements about linear independence. Lang conjectured [26, p. 100] that, under the assumptions of the Siegel–Shidlovskii Theorem, the linear independence of $f_1, \dots, f_m$ over $\overline{\mathbb{Q}}(z)$ should imply that of the numbers $f_1(\alpha), \dots, f_m(\alpha)$ over $\overline{\mathbb{Q}}$. After partial results of Shidlovskii [38, 39, 40, 42] and of Nesterenko and Shidlovskii [29], this was finally settled by Beukers [12], who proved the more precise lifting theorem: every homogeneous algebraic relation among $f_1(\alpha), \dots, f_m(\alpha)$ is obtained by specializing a homogeneous algebraic relation over $\overline{\mathbb{Q}}[z]$ among $f_1, \dots, f_m$.

(iii) The quantitative theory, namely the search for algebraic independence measures for values of E-functions. This direction goes back to Siegel [44]. The first general measure, under the assumption that $f_1, \dots, f_m$ are algebraically independent, was obtained by Lang in 1962 [25]. It was subsequently refined by various authors. Shidlovskii also pursued the more intricate problem of obtaining such measures for an algebraically independent subfamily of $\{f_1(\alpha), \dots, f_m(\alpha)\}$, without assuming the algebraic independence of $f_1, \dots, f_m$. See Section 1.1 for further details.

At the end of the 1990s, André [6, 7] introduced a new perspective based on the links between E- and G-functions revealed by the Borel and Laplace transforms within the theory of arithmetic Gevrey series. For our purposes, the main consequence of his work is the following; see [6, Th. 4.3, i)].

(A1) *The only possible non-trivial singularities of the minimal differential operator annihilating an E-function are 0 and ∞.*

One of the most striking aspects of André's approach is that, once (A1) is established, the Siegel–Shidlovskii Theorem follows from a comparatively short and essentially qualitative argument. This contrasts sharply with the classical proofs in transcendence theory, which rely on delicate auxiliary constructions and highly quantitative estimates. The quantitative core of the argument has not disappeared, however, but has been transferred to the proof of (A1), which rests on the theory of G-functions and, in particular, on a deep Diophantine result due to the Chudnovskys [16, Thm. VIII.1.5]: the minimal differential equation satisfied by a G-function satisfies the so-called Galochkin condition. One then shows that any differential equation satisfying Galochkin's condition is regular singular at 0. This can be deduced, as presented in [16], by combining results of André and Katz, or proved more directly by an argument of Beukers (see [27, Prop. 6.1]). Using the Laplace transform, André [6] was then able to deduce (A1) from these properties of G-operators. Until the present work, this was the only known strategy for proving (A1).

The importance of (A1) was further confirmed by Beukers' proof of the lifting theorem [12], which relies crucially on this result, thereby settling the problem described in Direction (ii). More generally, (A1) makes it possible to derive statements concerning *arbitrary values* of E-functions at algebraic points, independently of the singularities of a differential system in which the functions are embedded; see Section 1.2 for further details. It also plays an essential role in Delaygue's result [15] on the Lindemann–Weierstrass theorem, and in the recent work of the first author with Bell and Delaygue [2, 3], where a new method is developed to prove the algebraic independence of a subfamily of $\{f_1(\alpha), \dots, f_m(\alpha)\}$, without first establishing the algebraic independence over $\overline{\mathbb{Q}}(z)$ of the whole family $f_1, \dots, f_m$, as in Direction (i).

Another important result in the theory of E-functions is the following primality statement. It follows directly from a theorem of André [8, Thm. 1.6.1(3)], although his result is formulated in the language of algebraic geometry.

(A2) *Under the assumptions of the Siegel–Shidlovskii Theorem, let* $\mathfrak{I}$ *be the homogeneous ideal of* $\overline{\mathbb{Q}}[z][X_1, \dots, X_m]$ *generated by all homogeneous algebraic relations over* $\overline{\mathbb{Q}}[z]$ *among* $f_1, \dots, f_m$*, and let*

$$\mathrm{ev}_\alpha : \overline{\mathbb{Q}}[z, X_1, \dots, X_m] \longrightarrow \overline{\mathbb{Q}}[X_1, \dots, X_m]$$

be the evaluation morphism defined by $\mathrm{ev}_\alpha(P) := P(\alpha, X_1, \dots, X_m)$*. Then* $\mathrm{ev}_\alpha(\mathfrak{I})$ *is a prime ideal of* $\overline{\mathbb{Q}}[X_1, \dots, X_m]$*.*

In [8], André used his generalized differential Galois theory to establish (A2) and thereby derive a new proof of the lifting theorem that does not rely on (A1). Indeed, Statement (A2), together with the Siegel–Shidlovskii theorem, yields the lifting theorem by elementary commutative algebra, essentially through a comparison of Krull dimensions. Earlier, Nesterenko and Shidlovskii [29, Thm. 1] had already obtained the lifting theorem under the assumption (A2) by adapting the classical Siegel–Shidlovskii argument. Shidlovskii also obtained several further results conditional on (A2) (see [41, 42, 43]).

In this article, we revisit the classical Siegel–Shidlovskii method from a new perspective, described in Section 2. Our central idea is to assign a direct role to functional relations in the construction, rather than attempting to avoid them. A single underlying argument, with only minor variations, then yields the Siegel–Shidlovskii Theorem, Beukers' lifting theorem, and a general algebraic independence measure. For the Siegel–Shidlovskii Theorem itself, this perspective brings no substantial simplification. For the lifting theorem, however, our perspective makes both the role of (A2) and its close connection with the proof of the Siegel–Shidlovskii Theorem particularly clear. It also leads to a substantially simpler implementation of the direct approach of Nesterenko and Shidlovskii, avoiding a technical reduction process that is no longer needed in our framework. The main gain occurs in the quantitative setting. We establish a general algebraic independence measure (see Theorem 1.1), which settles the problem pursued by Shidlovskii in Direction (iii). Combined with (A1), it yields an algebraic independence measure of the same general type as those of Siegel and Lang for any algebraically independent family of E-values; see Corollary 1.2. A further variation of the method, still based on assigning a central role to functional relations, yields a proof of (A1) entirely within the theory of E-functions. In Appendix I,

we also give a short elementary proof of (A2), avoiding any appeal to differential Galois theory, either in its classical or generalized form.

Thus, apart from Siegel's classical construction based on Padé approximants and Shidlovskii's lemma, both of which are taken for granted here, the present paper recovers all the fundamental results of the transcendental theory of E-functions within a single self-contained framework.

1.1. **Shidlovskii's problem on algebraic independence measures.** In the first part of his 1929 memoir, Siegel applied his method to the Bessel function

$$J_0(z) := \sum_{n\geq 0} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}.$$

His primary aim was to prove that $J_0(\alpha)$ is transcendental for every non-zero algebraic number α, but his main result, the *Hauptsatz* of [44], was already quantitative. More precisely, if α is a non-zero algebraic number of degree δ, and if $P \in \mathbb{Z}[X,Y]$ is a non-zero polynomial of degree at most d and height at most H, then

$$\big|P\big(J_0(\alpha), J_0'(\alpha)\big)\big| > c\, H^{-123\delta^3 d^2},$$

where $c > 0$ depends only on α and d. In particular, $J_0(\alpha)$ and $J_0'(\alpha)$ are algebraically independent over $\overline{\mathbb{Q}}$.

The first general algebraic independence measure in the setting of a differential system was obtained by Lang in 1962 [25]. Under the assumptions of the Siegel–Shidlovskii Theorem, and assuming moreover that $f_1, \dots, f_m$ are algebraically independent over $\overline{\mathbb{Q}}(z)$, Lang proved that

$$\big|P\big(f_1(\alpha), \dots, f_m(\alpha)\big)\big| \geq C_1 H^{-C_2 d^m} \tag{1.2}$$

for every non-zero polynomial $P \in \mathbb{Z}[X_1, \dots, X_m]$ of degree at most d and height at most H. Here $C_2 > 0$ depends only on $f_1, \dots, f_m$ and α, while, for every $d \geq 1$, the constant $C_1 > 0$ depends only on d, $f_1, \dots, f_m$, and α, and may be chosen uniformly for all polynomials of degree at most d. The constants in this measure were subsequently refined by several authors [22, 28, 32, 35, 14, 23].

Shidlovskii devoted considerable effort to the more difficult problem of studying the algebraic independence of arbitrary subfamilies

$$\{f_{i_1}, \dots, f_{i_r}\} \subset \{f_1, \dots, f_m\} \tag{1.3}$$

and obtaining an algebraic independence measure whenever the corresponding values are algebraically independent; see [37, Chaps. 11–13] and [33, 34, 35, 36]. In his last published paper [43], he considered the still more general case in which the values need not be algebraically independent, and formulated the desired estimate in its natural alternative form: a polynomial in $f_1(\alpha), \dots, f_m(\alpha)$ either vanishes or its absolute value satisfies a lower bound of the form (1.2). He obtained only a partial result in this direction.

Our first main result settles Shidlovskii's problem.

Theorem 1.1. *Under the assumptions of the Siegel–Shidlovskii Theorem and with the notation of (1.3), the following alternative holds for every polynomial $P \in \mathbb{Z}[X_1, \dots, X_r]$:*

$$\begin{aligned} &\textit{either} \quad P\big(f_{i_1}(\alpha), \dots, f_{i_r}(\alpha)\big) = 0, \\ &\textit{or its absolute value satisfies a lower bound of the form (1.2),} \end{aligned} \tag{1.4}$$

with the exponent m replaced by $t := \mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}\big(f_1(z), \dots, f_m(z)\big) \le m$.

The vanishing alternative is essential since, in this generality, the corresponding values may well be algebraically dependent. When $f_{i_1}(\alpha), \dots, f_{i_r}(\alpha)$ are algebraically independent, however, Theorem 1.1 becomes an algebraic independence measure in the usual sense.

1.2. **André's theorem and arbitrary values of E-functions.** Beyond its crucial role in Beukers' proof of the lifting theorem [12], Theorem (A1) has been the starting point of many recent developments concerning arbitrary values of E-functions at algebraic points. Let

$$\mathbf{E} := \{f(\alpha) : f \text{ is an } E\text{-function}, \ \alpha \in \overline{\mathbb{Q}}\}$$

denote the ring of values of E-functions at algebraic points and, for a subfield $\mathbb{K} \subset \overline{\mathbb{Q}}$, let $\mathbf{E}_{\mathbb{K}}$ denote the set of values $f(\alpha)$ with $f \in \mathbb{K}[[z]]$ and $\alpha \in \mathbb{K}$. Any statement applying uniformly to all elements of $\mathbf{E}$, or of $\mathbf{E}_{\mathbb{K}}$, necessarily requires a means of overcoming the possible singularities of the differential systems involved; this is precisely what (A1) provides. We list below some of the results made possible by this theorem, notably in the work of Fischler, Rivoal, and the authors.

- Elements of $\mathbf{E}_{\mathbb{K}}$ are linearly dependent over $\overline{\mathbb{Q}}$ if and only if they are linearly dependent over $\mathbb{K}$. More precisely, if $\mathbb{K} \subset \mathbb{L} \subset \overline{\mathbb{Q}}$ are fields, then the canonical map $\mathbf{E}_{\mathbb{K}} \otimes_{\mathbb{K}} \mathbb{L} \to \mathbf{E}_{\mathbb{L}}$, $\xi \otimes \alpha \mapsto \alpha\xi$, is an isomorphism [4, 18, 20].
- There is an algorithm to determine all the algebraic values taken by a given E-function at algebraic points. More generally, one can decide linear dependence among elements of $\mathbf{E}$ and, modulo the computation of the corresponding functional ideal, determine all their algebraic relations [4, 5, 13, 19].
- Every algebraic relation among elements of $\mathbf{E}$ has a functional origin: it is obtained by specializing an algebraic differential relation among the functions involved [4].
- Non-zero linear forms in elements of $\mathbf{E}$ satisfy quantitative lower bounds. In particular, no element of $\mathbf{E}$ is a Liouville number, and every irrational element of $\mathbf{E}_{\mathbb{Q}}$ has irrationality exponent equal to 2 [20, 21].

Combining André's theorem (A1) with Theorem 1.1, we obtain the following intrinsic result.

Corollary 1.2. *Let $\xi_1, \dots, \xi_r \in \mathbf{E}$ be algebraically independent. Then, for every non-zero polynomial $P \in \mathbb{Z}[X_1, \dots, X_r]$, the quantity $|P(\xi_1, \dots, \xi_r)|$ satisfies a lower bound of the form (1.2), with the exponent m replaced by some positive integer t independent of P.*

Thus, as soon as values of E-functions are known to be algebraically independent, they *automatically* satisfy an algebraic independence measure of Siegel–Lang type. As shown in Section 4.3, the exponent t may moreover be bounded in terms of the transcendence degree of the differential field generated by the E-functions used to represent $\xi_1, \dots, \xi_r$. The fact that no element of $\mathbf{E}$ is a Liouville number also follows immediately from Corollary 1.2.

1.3. **Organization of the paper.** In Section 2, we recall the classical Siegel–Shidlovskii method and present our new perspective. Section 3 deals with values of E-functions at regular points of differential systems. There we give direct proofs of the Siegel–Shidlovskii Theorem and Beukers' lifting theorem, and establish the general algebraic independence measure of Theorem 1.1. In Section 4, we extend the method to points that are singular for the differential system under consideration and obtain a new proof of André's theorem (A1). We then deduce Corollary 1.2. Appendix I is devoted to a proof of André's result (A2). Finally, in Appendix II, we discuss consequences of our approach for the local structure of minimal differential equations annihilating E- and G-functions.

1.4. **Notation and definitions.** We let $\overline{\mathbb{Q}}$ denote the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Given a number field $\mathbb{K} \subset \overline{\mathbb{Q}}$, we let $\mathcal{O}_{\mathbb{K}}$ denote its ring of integers. The house of an algebraic number α is the maximum modulus of its Galois conjugates and is denoted by $\lceil \alpha \rceil$. The house of a linear form L with algebraic coefficients, denoted by $\lceil L \rceil$, is the maximum of the houses of its coefficients. We recall that a positive integer d is called a denominator of an algebraic number α if $d\alpha$ is an algebraic integer. As usual in the theory of E-functions, and throughout this paper, denominators and houses are used to measure the arithmetic complexity of algebraic numbers and linear forms, playing the role more commonly assigned to height functions. To align with standard notation, given a polynomial $P \in \mathbb{Z}[X_1, \dots, X_m]$, we let $H(P)$ denote its height, that is, the maximum of the absolute values of its coefficients. In this setting, $H(P)$ coincides with $\lceil P \rceil$ when P is viewed as a linear form in its monomials.

Given a linear differential equation

$$a_m(z)Y^{(m)} + a_{m-1}(z)Y^{(m-1)} + \cdots + a_0(z)Y = 0,$$

with coefficients in $\mathbb{C}[z]$, a point $\alpha \in \mathbb{C}$ is *singular* if it is a zero of the polynomial $a_m(z)$. A point that is not singular is called a *regular* or an *ordinary* point. A *trivial* or *apparent* singularity is a singular point at which the equation still admits a local basis of holomorphic solutions. A singular point α is *regular singular* if, for every $i \in \{1, \dots, m\}$, the rational function $(z-\alpha)^i a_{m-i}(z)/a_m(z)$ is holomorphic at α. A singularity which is not regular singular is called *irregular*.

In the case of a linear system

$$Y' = A(z)Y, \qquad A(z) \in \mathcal{M}_m(\mathbb{C}(z)),$$

a point $\alpha \in \mathbb{C}$ is singular if it is a pole of at least one of the entries of $A(z)$, and regular or ordinary otherwise.

Finally, recall that a power series

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$$

is an E-function if it satisfies a linear differential equation over $\overline{\mathbb{Q}}(z)$ and if there exists $C > 0$ such that $\lceil a_n \rceil \le C^n$ and $d_n \le C^n$ for all positive integers n, where d_n is the least common denominator of $a_0, \dots, a_n$. The field generated by the coefficients a_n, $n \ge 0$, of an E-function is a number field.

2. The Siegel–Shidlovskii method

The purpose of this section is to recall the main steps of the Siegel–Shidlovskii method and then to present our new perspective.

2.1. **Siegel's Diophantine argument.** The first objective of the method is to prove the linear independence of complex numbers, say $\xi_1, \dots, \xi_\nu$, or at least to obtain a lower bound for their linear rank. By viewing algebraic relations as linear forms in monomials, this framework naturally extends to algebraic independence and lower bounds on the transcendence degree. Natural refinements also concern linear and algebraic independence measures.

In each case, one argues by contradiction. The failure of the desired conclusion gives rise, for some positive integer η, to linear forms $L_1, \dots, L_\eta$ with coefficients in $\mathcal{O}_\mathbb{K}$ for some number field $\mathbb{K}$, linearly independent over $\mathbb{K}$, and taking *zero or exceptionally small values* at

$$\boldsymbol{\xi} := {}^{\mathrm{t}}(\xi_1, \dots, \xi_\nu) \in \mathbb{C}^\nu \setminus \{0\}.$$

We call $L_1, \dots, L_\eta$ the *putative forms*.

Independently of these forms, the method constructs, for large values of an auxiliary parameter n, a basis of linear forms in ν variables with coefficients of controlled arithmetic complexity and taking small values at $\boldsymbol{\xi}$. Their coefficient vectors form an invertible $\nu \times \nu$ matrix M_n with coefficients in $\mathcal{O}_\mathbb{K}$. By construction, $M_n\boldsymbol{\xi} = \mathbf{e}_n$, where $\mathbf{e}_n$ is a vector with *small norm*. We call this matrix a *preparation matrix*.

One then replaces suitable rows of M_n by the coefficient vectors of the putative forms while preserving invertibility. The resulting matrix $M_n' \in \mathcal{M}_\nu(\mathcal{O}_\mathbb{K})$ satisfies $M_n'\boldsymbol{\xi} = \boldsymbol{\varepsilon}_n$, where some coordinates of $\boldsymbol{\varepsilon}_n$ are much smaller than the corresponding ones in $\mathbf{e}_n$. Since M_n' is invertible, one has

$$\boldsymbol{\xi} = (M_n')^{-1}\boldsymbol{\varepsilon}_n. \tag{2.1}$$

By the comatrix formula, we have

$$(M_n')^{-1} = \frac{1}{\det M_n'}\,{}^{\mathrm{t}}\mathrm{Com}(M_n'),$$

where the entries of $\mathrm{Com}(M_n')$ are signed $(\nu-1)\times(\nu-1)$ minors of M_n'. The upper bounds for the entries of M_n' therefore control these minors. Moreover, $\det M_n'$ is a non-zero algebraic integer, so Liouville's inequality gives a lower bound for $|\det M_n'|$ in terms of its house. Combining these estimates yields, for each $j \in \{1, \dots, \nu\}$, a bound

$$\max_{1\le i\le \nu} \left|((M_n')^{-1})_{ij}\right| \le C_{j,n},$$

where the quantity $C_{j,n}$ may depend on the column. Writing

$$\boldsymbol{\varepsilon}_n := {}^{t}(\varepsilon_{1,n}, \dots, \varepsilon_{\nu,n}),$$

one obtains

$$\|(M_n')^{-1}\boldsymbol{\varepsilon}_n\|_\infty \le \sum_{j=1}^{\nu} C_{j,n}|\varepsilon_{j,n}|.$$

Since $\boldsymbol{\xi} \neq 0$, if, for a suitable choice of n, the values of the putative forms make the right-hand side sufficiently small, one obtains

$$\|(M_n')^{-1}\boldsymbol{\varepsilon}_n\|_\infty < \|\boldsymbol{\xi}\|_\infty,$$

in contradiction with (2.1). We call a matrix M_n' for which this contradiction occurs a *forbidden matrix*.

The method can therefore be summarized in three steps. One first constructs a sequence of preparation matrices $(M_n)_n$, then replaces some of their rows by coefficient vectors of putative forms, and finally proves that, for a suitable n, the resulting matrix M_n' is a forbidden matrix. The essential difficulty lies in the first step.

Remark. The preparation matrices M_n need not be invertible; it is enough that the matrices M_n' are. We shall use this when proving André's theorem (A1).

2.2. **Classical construction of the preparation matrix.** Let $f_1, \dots, f_m$ be E-functions which are linearly independent over $\overline{\mathbb{Q}}(z)$ and satisfy

$$Y'(z) = A(z)Y(z), \qquad A(z) \in \mathcal{M}_m(\overline{\mathbb{Q}}(z)).$$

Let $\alpha \in \overline{\mathbb{Q}}^\times$ be a regular point of this system, and set

$$\boldsymbol{\xi} := {}^{\mathrm{t}}(f_1(\alpha), \dots, f_m(\alpha)).$$

We fix a number field $\mathbb{K}$ containing α, the coefficients of $A(z)$, and all the algebraic coefficients occurring in the construction.

The classical preparation matrix is obtained by combining Siegel's lemma and Shidlovskii's lemma. Siegel's lemma, applied over $\mathbb{K}$, provides polynomials $P_{1,n}(z), \dots, P_{m,n}(z) \in \mathcal{O}_\mathbb{K}[z]$ of controlled degree, denominator, and house such that the functional linear form

$$R_n(z) := P_{1,n}(z)f_1(z) + \cdots + P_{m,n}(z)f_m(z)$$

has a large order of vanishing at the origin.

The differential system allows one to differentiate R_n and, after clearing the denominators, to rewrite its successive derivatives in the form

$$R_{n,k}(z) := P_{1,n,k}(z)f_1(z) + \cdots + P_{m,n,k}(z)f_m(z),$$

where the polynomials $P_{i,n,k}$ have coefficients in $\mathcal{O}_\mathbb{K}$ and controlled degree and house.

Shidlovskii's lemma ensures that, for a second auxiliary parameter $\varepsilon > 0$, which may be chosen arbitrarily small, one can select m such forms among those with indices $1 \leq k \leq \varepsilon n + C$ whose specialized coefficient vectors at $z = \alpha$ are linearly independent. Here C is independent of n. This bound on the indices guarantees that their coefficients remain controlled and that their values at $\boldsymbol{\xi}$ remain small. These vectors form the rows of an invertible preparation matrix $M_n \in \mathcal{M}_m(\mathbb{K})$.

Two fundamental restrictions are inherent in this construction:

(1) the functions must be linearly independent over $\overline{\mathbb{Q}}(z)$;
(2) the point α must be regular for the differential system.

When $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K}$ is an imaginary quadratic field, the computation shows that one single putative form is sufficient to obtain a forbidden matrix. The linear independence of $f_1(\alpha), \dots, f_m(\alpha)$ over $\mathbb{K}$ follows. If the functions $f_1, \dots, f_m$ are algebraically independent over $\overline{\mathbb{Q}}(z)$, one deduces the algebraic independence of

$f_1(\alpha), \dots, f_m(\alpha)$ over $\overline{\mathbb{Q}}$ by applying the same argument to the associated monomials and using their linear independence. For general number fields, however, more than one putative form is needed, and the contradiction only yields a lower bound for the $\mathbb{K}$-rank of $f_1(\alpha), \dots, f_m(\alpha)$. But one can still recover the algebraic independence of $f_1(\alpha), \dots, f_m(\alpha)$ by applying such a lower bound asymptotically to monomials of large degree.

2.2.1. *Obstruction arising from functional relations.* When $f_1, \dots, f_m$ are linearly dependent, the classical remedy is to choose a basis of their span and to apply the method to the induced differential system, thus reducing the dimension. This reduction changes the natural coordinates of the problem. More precisely, the preparation matrix is no longer constructed from the original family of functions, but from a basis of its $\overline{\mathbb{Q}}(z)$-span. Thus, if the initial family consists of m functions of rank $r < m$, the preparation matrix has size $r \times r$ rather than $m \times m$.

The same phenomenon becomes much more pronounced when studying algebraic relations. There, one starts with the monomials of degree δ in the functions $f_1, \dots, f_m$, but the preparation matrix is again constructed only after replacing this natural family by a basis of its $\overline{\mathbb{Q}}(z)$-span. The putative forms, however, are naturally expressed in terms of the original monomials. One must therefore rewrite them in the chosen basis and control their independence throughout this change of coordinates. This reduction process is a major source of technical complications and has proved to be a serious obstacle in several situations.

2.3. **The new construction.** The key difference with the classical method lies in the treatment of functional relations. Instead of eliminating them before constructing the preparation matrix, our approach incorporates them into the preparation matrix itself.

2.3.1. *The construction at regular points.* Let $g_1, \dots, g_\nu \in \mathbb{K}[[z]]$ be E-functions which satisfy

$$Y'(z) = B(z)Y(z), \qquad B(z) \in \mathcal{M}_\nu(\overline{\mathbb{Q}}(z)).$$

Let $\alpha \in \overline{\mathbb{Q}}^\times$ be a regular point of this system. One can think of $g_1, \dots, g_\nu$ as monomials in some algebraically dependent E-functions $f_1, \dots, f_m$ one starts with, so that $g_1, \dots, g_\nu$ may not be linearly independent.

Let μ denote the rank of $g_1, \dots, g_\nu$ over $\mathbb{K}(z)$. The space of functional linear relations among them has dimension $\nu - \mu$ over $\mathbb{K}(z)$. We choose a basis of this space whose coefficients are regular at $z = \alpha$ and whose specializations at α remain linearly independent and have coefficients in $\mathcal{O}_\mathbb{K}$. We show that the Padé construction, applied to the g_i, produces μ independent forms in directions transverse to these relations, and we complete them by adjoining the relations themselves.

After specialization at $z = \alpha$, the preparation matrix M_n has the form

$$\left(\frac{\mu \text{ Padé-type forms depending on } n}{\nu - \mu \text{ specialized functional relations}}\right) \in \mathcal{M}_\nu(\mathcal{O}_\mathbb{K}),$$

and

$$M_n \boldsymbol{\xi} = {}^{\mathrm{t}}(e_{1,n}, \dots, e_{\mu,n}, 0, \dots, 0), \qquad \boldsymbol{\xi} := {}^{\mathrm{t}}(g_1(\alpha), \dots, g_\nu(\alpha)).$$

The functional relations supply exactly the rows missing from the Padé construction. Their coefficients are independent of n, and their values at $\boldsymbol{\xi}$ vanish

identically. In this sense, the forms arising from functional relations are far better than those produced by Padé approximation and differentiation. What constitutes an obstruction in the classical construction becomes a source of exceptionally good rows in the new one.

In the proofs of our main results, we show that the negation of the desired conclusion provides putative forms whose coefficient vectors can be substituted for Padé-type rows. Since all the forms remain expressed in the original coordinates, their independence is preserved, and the resulting matrix is a forbidden matrix.

This principle is the key to the proof of Theorem 1.1. It also clarifies and unifies the arguments: Theorem 1.1, the Siegel–Shidlovskii Theorem, and Beukers' lifting theorem all follow from essentially the same mechanism. The preparation matrices and the forbidden-matrix argument remain unchanged; only the putative forms and the final conclusion differ.

2.3.2. *The construction at singular points.* The same principle makes it possible to work at singular points.

At a regular point, the successive derivatives of a Padé approximant can be rewritten as linear forms in the original functions, with coefficients regular at the point of specialization. At a singular point, the differential system introduces poles and therefore no longer directly produces a preparation matrix.

The key idea is therefore to avoid this rewriting procedure by simply enlarging the family of functions, adjoining the derivatives that occur in the Padé construction. The resulting family is linearly dependent over $\mathbb{K}(z)$ and would therefore fall outside the scope of the classical method. In our approach, however, the functional relations simply provide additional rows of a higher-dimensional preparation matrix.

Suppose that an E-function f satisfies a linear differential equation of order ℓ which is singular at α. Then the corresponding relation among $f, f', \dots, f^{(\ell)}$ specializes at $z = \alpha$ to a non-trivial relation of smaller order, involving only

$$f(\alpha), \dots, f^{(\ell-1)}(\alpha).$$

In particular, if f satisfies no linear differential equation which is regular at α, then, after adjoining sufficiently many derivatives, one obtains too many specialized relations of smaller order to be accommodated by the preparation matrix. Inserting these relations as rows ultimately produces a forbidden matrix. This yields our proof of André's theorem (A1).

3. Applying the method at regular points

In this section, we consider values of E-functions related by a differential system at a non-zero algebraic point which is regular for this system. We prove the Siegel–Shidlovskii Theorem, Beukers' lifting theorem, and Theorem 1.1.

All three proofs rely on the same sequence of preparation matrices. They only differ in the construction of the putative forms. In the proofs of the Siegel–Shidlovskii and lifting theorems, the putative forms vanish at the point under consideration. In the proof of the algebraic independence measure, they are non-zero but extremely small. This requires a more delicate adjustment of the auxiliary parameters and makes the quantitative argument somewhat longer, whereas the underlying matrix argument remains the same.

3.1. **Notation.** Throughout this section, let $\mathbb{K}$ be a number field, let $f_1, \dots, f_m \in \mathbb{K}[[z]]$ be E-functions satisfying a differential system

$$Y'(z) = A(z)Y(z), \qquad A(z) \in \mathcal{M}_m(\mathbb{K}(z)),$$

and let $\alpha \in \mathbb{K}^\times$ be a point which is not a pole of A. Since the class of E-functions is invariant under the change of variable $z \mapsto \alpha z$, we may assume that $\alpha = 1$.

We denote by $\mathfrak{I}$ the ideal of $\mathbb{K}[z, X_1, \dots, X_m]$ generated by all polynomials $Q(z, X_1, \dots, X_m)$ that are homogeneous in $X_1, \dots, X_m$ and satisfy

$$Q(z, f_1(z), \dots, f_m(z)) = 0.$$

Thus $\mathfrak{I}$ is a homogeneous ideal with respect to the variables $X_1, \dots, X_m$. We further set

$$\mathfrak{I}_{\mathbb{K}(z)} := \mathbb{K}(z)\mathfrak{I} \subseteq \mathbb{K}(z)[X_1, \dots, X_m].$$

Then $\mathfrak{I}_{\mathbb{K}(z)}$ is precisely the ideal generated by the homogeneous algebraic relations over $\mathbb{K}(z)$ among $f_1, \dots, f_m$.

We recall that ev_1 denotes the specialization map

$$\mathrm{ev}_1 : \begin{cases} \mathbb{K}[z, X_1, \dots, X_m] \longrightarrow \mathbb{K}[X_1, \dots, X_m], \\ Q(z, X_1, \dots, X_m) \longmapsto Q(1, X_1, \dots, X_m). \end{cases}$$

Finally, let $\mathfrak{I}(1)$ denote the ideal of $\mathbb{K}[X_1, \dots, X_m]$ generated by all homogeneous polynomials $P(X_1, \dots, X_m)$ such that

$$P(f_1(1), \dots, f_m(1)) = 0.$$

Thus $\mathfrak{I}(1)$ is homogeneous, and we have $\mathrm{ev}_1(\mathfrak{I}) \subseteq \mathfrak{I}(1)$.

Given an ideal $\mathfrak{a}$ homogeneous with respect to $X_1, \dots, X_m$ and an integer $\delta \geq 0$, we denote by $\mathfrak{a}_\delta$ its homogeneous component of degree δ in these variables.

3.2. **The sequence of preparation matrices.** Let $\delta \geq 1$ be an integer, to be chosen later. We consider the monomials of total degree δ in the variables $X_1, \dots, X_m$. Let $Y_1, \dots, Y_\nu$ be an enumeration of these monomials, and let $g_1, \dots, g_\nu$ be the corresponding monomials in $f_1, \dots, f_m$. We set

$$\boldsymbol{\xi} := {}^{\mathrm{t}}(g_1(1), \dots, g_\nu(1)), \qquad \boldsymbol{X} := (X_1, \dots, X_m), \qquad \boldsymbol{Y} := (Y_1, \dots, Y_\nu),$$

keeping in mind that each Y_i is a monomial of total degree δ in $\boldsymbol{X}$.

3.2.1. *Forms arising from functional relations.* Let μ denote the dimension over $\mathbb{K}(z)$ of the vector space spanned by $g_1, \dots, g_\nu$. The kernel of the map

$$\mathbb{K}(z)Y_1 \oplus \cdots \oplus \mathbb{K}(z)Y_\nu \longrightarrow \mathbb{K}(z)g_1 + \cdots + \mathbb{K}(z)g_\nu, \qquad Y_i \longmapsto g_i,$$

identifies with $(\mathfrak{I}_{\mathbb{K}(z)})_\delta$. Hence

$$\dim_{\mathbb{K}(z)}(\mathfrak{I}_{\mathbb{K}(z)})_\delta = \nu - \mu.$$

Thus $\nu - \mu$ is the number of independent homogeneous functional relations of degree δ, whereas μ is the number of additional Padé forms needed to complete the preparation matrix.

We may choose forms

$$\ell_1(z, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(z, \boldsymbol{Y}) \in \mathfrak{I}_\delta$$

with coefficients in $\mathcal{O}_{\mathbb{K}}[z]$ which form a basis of $(\mathfrak{I}_{\mathbb{K}(z)})_\delta$ over $\mathbb{K}(z)$ and whose specializations

$$\ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y})$$

are linearly independent over $\mathbb{K}$; see, for instance, [12, Lem. 3.1]. In particular,

$$\ell_1(1, \boldsymbol{\xi}) = \cdots = \ell_{\nu-\mu}(1, \boldsymbol{\xi}) = 0.$$

3.2.2. *Forms arising from Padé approximants.* We now complete the family

$$\ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y})$$

by linear forms taking small values at $\boldsymbol{\xi}$. We use the following classical consequence of Siegel's construction and Shidlovskii's lemma.

Proposition 3.1. *Let s be a positive integer, and let $h_1, \dots, h_s \in \mathbb{K}[[z]]$ be linearly independent E-functions related by a linear differential system of size s. Let $0 < \varepsilon_0 < 1/2$. There exists a positive real number ρ, depending only on $h_1, \dots, h_s$ and ε_0, such that, for every positive integer n, there exist polynomials $P_1, \dots, P_s \in \mathcal{O}_{\mathbb{K}}[z]$, not all zero, of degree at most n and house at most $\rho n^{(1+\varepsilon_0)n}$, such that*

$$\operatorname{val} R(z) \geq s(n+1) - \varepsilon_0 n - 1,$$

where

$$R(z) := P_1(z)h_1(z) + \cdots + P_s(z)h_s(z).$$

Furthermore, there exists a positive integer n_0, depending only on $h_1, \dots, h_s$, such that $R(z), R'(z), \dots, R^{(s-1)}(z)$ are linearly independent over $\mathbb{K}(z)$ whenever $n \geq \max\{n_0, \varepsilon_0^{-1}\}$.

Proof. The existence of $P_1, \dots, P_s$, not all zero, satisfying the stated degree, height, and vanishing estimates follows directly from Siegel's lemma; see [37, Chap. 3, Lem. 14]. Since $\varepsilon_0 < 1$, the linear independence assertion follows immediately from Shidlovskii's lemma; see [17, Lem. 5.2] or [37, Chap. 3, Lem. 8]. □

The $\mathbb{K}(z)$-vector space spanned by $g_1, \dots, g_\nu$ has dimension μ. After reindexing, we may assume that $g_1, \dots, g_\mu$ form a basis of this space. Since $f_1, \dots, f_m$ satisfy a linear differential system, the same holds for $g_1, \dots, g_\nu$, and hence for $g_1, \dots, g_\mu$.

Let $0 < \varepsilon < 1$ be sufficiently small, to be chosen later. We apply Proposition 3.1 with

$$s := \mu, \qquad \varepsilon_0 := \frac{\varepsilon}{6(\mu+1)}, \qquad h_i := g_i \quad (1 \leq i \leq \mu).$$

Set $n_1 := \max\{n_0, \varepsilon_0^{-1}\}$. For every $n \geq n_1$, let $P_1, \dots, P_\mu$ and R be provided by Proposition 3.1. Since $z = 1$ is regular for the differential system satisfied by $f_1, \dots, f_m$, the functions $g_1, \dots, g_\nu$ satisfy a differential system

$$Y'(z) = B(z)Y(z)$$

which is regular at $z = 1$. Let $T(z) \in \mathcal{O}_{\mathbb{K}}[z]$ be a least common denominator of the entries of $B(z)$, chosen so that $T(z)B(z)$ has entries in $\mathcal{O}_{\mathbb{K}}[z]$ and $T(1) \neq 0$.

Let $\partial = d/dz$. For every linear form $L(z, \boldsymbol{Y}) \in \mathcal{O}_{\mathbb{K}}[z, \boldsymbol{Y}]$, there exist polynomials $B_1(z), \dots, B_\nu(z) \in \mathcal{O}_{\mathbb{K}}[z]$ such that

$$T(z)\partial\big(L(z, \boldsymbol{g}(z))\big) = \sum_{i=1}^{\nu} B_i(z)g_i(z).$$

We define

$$\Theta(L)(z,\boldsymbol{Y}) := \sum_{i=1}^{\nu} B_i(z)Y_i.$$

Thus $\Theta(L)(z,\boldsymbol{g}(z)) = T(z)\partial\big(L(z,\boldsymbol{g}(z))\big)$. We then define inductively

$$R_0(z,\boldsymbol{Y}) := P_1(z)Y_1 + \cdots + P_\mu(z)Y_\mu, \qquad R_{k+1}(z,\boldsymbol{Y}) := \Theta(R_k)(z,\boldsymbol{Y}).$$

Notice that R_0 is supported on $Y_1,\dots,Y_\mu$, whereas the forms R_k, for $k \geq 1$, may involve all the variables $Y_1,\dots,Y_\nu$.

Lemma 3.2. *There exist positive constants c_1,ρ_1,ρ_2, independent of n, such that, for every $n \geq n_1$, there exists an integer c_2 satisfying $0 \leq c_2 \leq \varepsilon_0 n + c_1$ and μ forms $L_1(\boldsymbol{Y}),\dots,L_\mu(\boldsymbol{Y})$ chosen among $R_0(1,\boldsymbol{Y}),\dots,R_{\mu+c_2-1}(1,\boldsymbol{Y})$ such that:*

(a) $\big|L_i(\boldsymbol{\xi})\big| \leq \rho_1 n^{-(\mu-1-\varepsilon)n}$ $\quad (1 \leq i \leq \mu)$;
(b) $\boxed{L_i} \leq \rho_2 n^{(1+\varepsilon)n}$ $\quad (1 \leq i \leq \mu)$;
(c) *the ν linear forms $L_1(\boldsymbol{Y}),\dots,L_\mu(\boldsymbol{Y}),\ell_1(1,\boldsymbol{Y}),\dots,\ell_{\nu-\mu}(1,\boldsymbol{Y})$ are linearly independent over $\mathbb{K}$.*

Proof. Once c_1 is fixed, the classical estimates show that there exist positive constants ρ_1,ρ_2, independent of n, such that every form $R_i(1,\boldsymbol{Y})$ with $0 \leq i \leq \varepsilon_0 n + c_1$ satisfies

$$\big|R_i(1,\boldsymbol{\xi})\big| \leq \rho_1 n^{-(\mu-1-\varepsilon)n} \quad \text{and} \quad \boxed{R_i(1,\boldsymbol{Y})} \leq \rho_2 n^{(1+\varepsilon)n}.$$

Detailed estimates may be found in [37, Chap. 3, §10]. Thus any forms chosen in the indicated range satisfy (a) and (b).

To prove (c), it is enough to find an integer $c_2 \leq \varepsilon_0 n + c_1$ such that the forms

$$R_0(1,\boldsymbol{Y}),\dots,R_{\mu+c_2-1}(1,\boldsymbol{Y}),\ell_1(1,\boldsymbol{Y}),\dots,\ell_{\nu-\mu}(1,\boldsymbol{Y})$$

have rank ν. We adapt Siegel's original argument [17, p. 219] to the presence of these two families of forms.

Consider the matrix

$$\Gamma(z) := (\gamma_{i,j}(z))_{1\leq i,j\leq \nu}$$

defined by

$$\gamma_{i,j}(z) := \begin{cases} \text{the coefficient of } Y_j \text{ in } R_{i-1}(z,\boldsymbol{Y}), & 1 \leq i \leq \mu, \\ \text{the coefficient of } Y_j \text{ in } \ell_{i-\mu}(z,\boldsymbol{Y}), & \mu < i \leq \nu. \end{cases}$$

It has entries in $\mathcal{O}_{\mathbb{K}}[z]$ and satisfies

$$\Gamma(z)\begin{pmatrix} g_1(z) \\ \vdots \\ g_\nu(z) \end{pmatrix} = \begin{pmatrix} R_0(z,\boldsymbol{g}(z)) \\ \vdots \\ R_{\mu-1}(z,\boldsymbol{g}(z)) \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \tag{3.1}$$

as well as

$$\Gamma(z)\begin{pmatrix} Y_1 \\ \vdots \\ Y_\nu \end{pmatrix} = \begin{pmatrix} R_0(z,\boldsymbol{Y}) \\ \vdots \\ R_{\mu-1}(z,\boldsymbol{Y}) \\ \ell_1(z,\boldsymbol{Y}) \\ \vdots \\ \ell_{\nu-\mu}(z,\boldsymbol{Y}) \end{pmatrix}.$$

Let us first prove that $\Gamma(z)$ is invertible. By construction, the functions

$$R_0(z,\boldsymbol{g}(z)),\dots,R_{\mu-1}(z,\boldsymbol{g}(z))$$

span the same $\mathbb{K}(z)$-vector space as

$$R(z),R'(z),\dots,R^{(\mu-1)}(z).$$

They are therefore linearly independent by Proposition 3.1. Moreover, the forms

$$\ell_1(z,\boldsymbol{Y}),\dots,\ell_{\nu-\mu}(z,\boldsymbol{Y})$$

are linearly independent and vanish at $\boldsymbol{Y}=\boldsymbol{g}(z)$. It follows that

$$R_0(z,\boldsymbol{Y}),\dots,R_{\mu-1}(z,\boldsymbol{Y}),\ell_1(z,\boldsymbol{Y}),\dots,\ell_{\nu-\mu}(z,\boldsymbol{Y})$$

are linearly independent over $\mathbb{K}(z)$. Hence $\Gamma(z)$ is invertible.

Set $\Delta(z) := \det\Gamma(z) \neq 0$. Since $P_1,\dots,P_\mu$ have degree at most n, whereas the degrees of the coefficients of the forms ℓ_i are bounded independently of n, there exists a constant χ_1, independent of n, such that $\deg\Delta \leq \mu n + \chi_1$.

On the other hand, by applying the comatrix formula to (3.1) and using

$$\operatorname{val}\big(R_k(z,\boldsymbol{g}(z))\big) \geq \mu(n+1) - \varepsilon_0 n - k - 1,$$

we find an index $i \in \{1,\dots,\nu\}$ such that

$$\operatorname{val}(\Delta) + \operatorname{val}(g_i) \geq \mu n - \varepsilon_0 n.$$

Since the quantities $\operatorname{val}(g_i)$ are bounded independently of n, it follows that the order of vanishing of Δ at $z=1$ is at most $\varepsilon_0 n + \chi_2$ for some constant χ_2 independent of n.

Let χ_3 be the smallest non-negative integer such that $\partial^{\chi_3}\Delta(1) \neq 0$. Then

$$\chi_3 \leq \varepsilon_0 n + \chi_2. \tag{3.2}$$

Using the comatrix formula for Γ, we obtain, for every $i \in \{1,\dots,\nu\}$,

$$\Delta(z)Y_i = \sum_{j=0}^{\mu-1} \Delta_{i,j+1}(z)R_j(z,\boldsymbol{Y}) + \sum_{j=1}^{\nu-\mu} \Delta_{i,j+\mu}(z)\ell_j(z,\boldsymbol{Y}),$$

where $\Delta_{i,j}(z)$ denotes the corresponding cofactor. Applying Θ successively χ_3 times and then specializing at $z=1$, all the terms involving $\partial^k\Delta(1)$ with $k<\chi_3$ vanish. Since $T(1)\neq 0$ and $\partial^{\chi_3}\Delta(1)\neq 0$, it follows that every Y_i belongs to the $\mathbb{K}$-vector space spanned by

$$R_j(1,\boldsymbol{Y}) \qquad (0 \leq j \leq \mu+\chi_3-1)$$

and

$$\ell_j^{[k]}(1,\boldsymbol{Y}) \qquad (1 \leq j \leq \nu-\mu,\ 0 \leq k \leq \chi_3),$$

where $\ell_j^{[0]} = \ell_j$ and $\ell_j^{[k+1]} = \Theta(\ell_j^{[k]})$. Consequently, these forms have rank ν.

For every j and k, one has

$$\ell_j^{[k]}(z, \boldsymbol{g}(z)) = (T\partial)^k\big(\ell_j(z, \boldsymbol{g}(z))\big) = (T\partial)^k\big(0\big) = 0.$$

Hence $\ell_j^{[k]} \in \mathfrak{I}_\delta$, and therefore $\ell_j^{[k]}(1, \boldsymbol{Y})$ belongs to $\mathrm{ev}_1(\mathfrak{I}_\delta)$. Since

$$\ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y})$$

form a basis of $\mathrm{ev}_1(\mathfrak{I}_\delta)$ over $\mathbb{K}$, it follows that every $\ell_j^{[k]}(1, \boldsymbol{Y})$ is a linear combination of these forms. Consequently,

$$R_0(1, \boldsymbol{Y}), \dots, R_{\mu+\chi_3-1}(1, \boldsymbol{Y}), \ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y})$$

have rank ν. In view of (3.2), the conclusion follows by taking $c_1 = \chi_2$ and $c_2 = \chi_3$. □

For every $n \geq n_1$, let $L_1, \dots, L_\mu$ be the forms provided by Lemma 3.2. We define the matrix

$$M_n := (a_{n;i,j})_{1 \leq i,j \leq \nu}$$

by setting

$$a_{n;i,j} := \begin{cases} \text{the coefficient of } Y_j \text{ in } L_i(\boldsymbol{Y}), & 1 \leq i \leq \mu, \\ \text{the coefficient of } Y_j \text{ in } \ell_{i-\mu}(1, \boldsymbol{Y}), & \mu < i \leq \nu. \end{cases}$$

By Lemma 3.2, the matrix M_n is invertible. Its first μ rows correspond to forms arising from the Padé approximants, whereas its remaining $\nu - \mu$ rows are the specializations at $z = 1$ of functional relations. The sequence $(M_n)_{n \geq n_1}$ is our sequence of preparation matrices.

3.3. **The Siegel–Shidlovskii theorem.** We are now ready to prove the Siegel–Shidlovskii theorem.

Theorem 3.3 (Siegel–Shidlovskii, 1959)**.** *Let $f_1, \dots, f_m$ be E-functions satisfying a linear differential system, and let $\alpha \in \overline{\mathbb{Q}}^\times$ be a regular point with respect to this system. Then*

$$\mathrm{tr.deg}_{\overline{\mathbb{Q}}}\big(f_1(\alpha), \dots, f_m(\alpha)\big) = \mathrm{tr.deg}_{\overline{\mathbb{Q}}(z)}\big(f_1(z), \dots, f_m(z)\big).$$

Proof. We continue with the notation of Sections 3.1 and 3.2. In particular, we assume that $\alpha = 1$. By adjoining the constant function 1 if necessary, we may also assume that $f_m = 1$. Set

$$t := \mathrm{tr.deg}_{\mathbb{K}(z)}\big(f_1(z), \dots, f_m(z)\big) \quad \text{and} \quad t_0 := \mathrm{tr.deg}_{\mathbb{K}}\big(f_1(1), \dots, f_m(1)\big).$$

Since $\mathbb{K}/\mathbb{Q}$ is algebraic, these are also the transcendence degrees appearing in the statement of the theorem. Assume, for a contradiction, that $t_0 < t$.

Since $f_m = 1$, homogenization with respect to X_m identifies $\mathfrak{I}(1)_\delta$ with the space of polynomials in $X_1, \dots, X_{m-1}$ of degree at most δ which vanish at

$$f_1(1), \dots, f_{m-1}(1),$$

and identifies $(\mathfrak{I}_{\mathbb{K}(z)})_\delta$ with the space of polynomials in $X_1, \dots, X_{m-1}$ of degree at most δ which vanish at

$$f_1(z), \dots, f_{m-1}(z).$$

It follows from the Hilbert–Serre theorem (see for instance [17, p. 231]) that, for all sufficiently large δ, the quantities

$$\kappa := \nu - \dim_{\mathbb{K}} \mathfrak{I}(1)_\delta \quad \text{and} \quad \mu = \nu - \dim_{\mathbb{K}(z)} (\mathfrak{I}_{\mathbb{K}(z)})_\delta$$

are given by polynomials in δ of degrees t_0 and t, respectively. Since $t_0 < t$, we may choose δ sufficiently large so that

$$\mu > 2[\mathbb{K} : \mathbb{Q}]\kappa. \tag{3.3}$$

By construction, $\mathrm{ev}_1(\mathfrak{I}_\delta) \subseteq \mathfrak{I}(1)_\delta$. Moreover, $\ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y})$ form a basis of $\mathrm{ev}_1(\mathfrak{I}_\delta)$ over $\mathbb{K}$. Since $\dim_{\mathbb{K}} \mathfrak{I}(1)_\delta = \nu - \kappa$, we may choose linear forms

$$N_1(\boldsymbol{Y}), \dots, N_\eta(\boldsymbol{Y}) \in \mathcal{O}_{\mathbb{K}}[\boldsymbol{Y}], \qquad \eta := \mu - \kappa,$$

such that

$$N_1(\boldsymbol{Y}), \dots, N_\eta(\boldsymbol{Y}), \ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y})$$

form a basis of $\mathfrak{I}(1)_\delta$ over $\mathbb{K}$.

On the other hand,

$$L_1(\boldsymbol{Y}), \dots, L_\mu(\boldsymbol{Y}), \ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y})$$

form a basis of the $\mathbb{K}$-vector space of all linear forms in $\boldsymbol{Y}$. The images of $L_1, \dots, L_\mu$ therefore span the quotient of this space by $\mathfrak{I}(1)_\delta$, which has dimension κ. After reindexing the forms L_i, we may consequently assume that

$$L_1(\boldsymbol{Y}), \dots, L_\kappa(\boldsymbol{Y}), N_1(\boldsymbol{Y}), \dots, N_\eta(\boldsymbol{Y}), \ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y}) \tag{3.4}$$

form a basis of the space of all linear forms in $\boldsymbol{Y}$.

Let $M_n' \in \mathcal{M}_\nu(\mathcal{O}_{\mathbb{K}})$ be the matrix whose rows are the coefficient vectors, with respect to $Y_1, \dots, Y_\nu$, of the forms in (3.4). This will be our forbidden matrix.

Since

$$N_i(\boldsymbol{\xi}) = 0 \qquad (1 \le i \le \eta), \quad \text{and} \quad \ell_i(1, \boldsymbol{\xi}) = 0 \qquad (1 \le i \le \nu - \mu),$$

we have

$$M_n' \boldsymbol{\xi} = {}^{\mathrm{t}}\big(L_1(\boldsymbol{\xi}), \dots, L_\kappa(\boldsymbol{\xi}), 0, \dots, 0\big) =: \boldsymbol{\varepsilon}_n. \tag{3.5}$$

By Lemma 3.2, $\|\boldsymbol{\varepsilon}_n\|_\infty \le \rho_1 n^{-(\mu-1-\varepsilon)n}$.

Let C_n denote the maximum of the absolute values of the entries in the first κ columns of $(M_n')^{-1}$. Since only the first κ coordinates of $\boldsymbol{\varepsilon}_n$ may be non-zero, we obtain

$$\|\boldsymbol{\xi}\|_\infty = \|(M_n')^{-1}\boldsymbol{\varepsilon}_n\|_\infty \le \kappa C_n \|\boldsymbol{\varepsilon}_n\|_\infty \le \kappa \rho_1 C_n n^{-(\mu-1-\varepsilon)n}. \tag{3.6}$$

We now estimate C_n using the comatrix formula. Set $H_n := \rho_2 n^{(1+\varepsilon)n}$. Then H_n is an upper bound for the houses of $L_1, \dots, L_\kappa$. Let H be an upper bound for the houses of the coefficients of

$$N_1, \dots, N_\eta, \ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y}).$$

These forms are independent of n, and hence so is H.

Since $\det M_n' \in \mathcal{O}_{\mathbb{K}} \setminus \{0\}$, Liouville's inequality gives

$$|\det M_n'| \ge \big|\overline{\det M_n'}\big|^{1-[\mathbb{K}:\mathbb{Q}]}.$$

Moreover,

$$\big|\overline{\det M_n'}\big| \le \nu! H_n^\kappa H^{\nu-\kappa}.$$

Consequently, there exists a positive constant ρ_3, independent of n, such that

$$|\det M'_n| \geq \rho_3 H_n^{-\kappa([\mathbb{K}:\mathbb{Q}]-1)}. \tag{3.7}$$

An entry in one of the first κ columns of $\mathrm{Com}(M'_n)^{\mathrm{t}}$ is a cofactor obtained by deleting one of the first κ rows of M'_n. It therefore involves at most $\kappa - 1$ rows whose coefficients have house bounded by H_n. Hence its absolute value is at most $(\nu-1)!H_n^{\kappa-1}H^{\nu-\kappa}$. Combining this estimate with (3.7), we obtain a positive constant ρ_4, independent of n, such that

$$C_n \leq \rho_4 H_n^{\kappa[\mathbb{K}:\mathbb{Q}]-1}. \tag{3.8}$$

Combining (3.6) and (3.8), and absorbing all constants independent of n into a positive constant ρ_5, we obtain

$$\begin{aligned} \|(M'_n)^{-1}\boldsymbol{\varepsilon}_n\|_\infty &\leq \rho_5 n^{(1+\varepsilon)n(\kappa[\mathbb{K}:\mathbb{Q}]-1)} n^{-(\mu-1-\varepsilon)n} \\ &= \rho_5 n^{((1+\varepsilon)\kappa[\mathbb{K}:\mathbb{Q}]-\mu)n}. \end{aligned} \tag{3.9}$$

By (3.3) and the assumption $0 < \varepsilon < 1$, we have $(1+\varepsilon)\kappa[\mathbb{K}:\mathbb{Q}] - \mu < 0$. The right-hand side of (3.9) therefore tends to zero as n tends to infinity. Since $\boldsymbol{\xi} \neq 0$, this contradicts (3.6). Thus $t_0 \geq t$. The reverse inequality is immediate by specialization of algebraic relations. We conclude that $t_0 = t$, which completes the proof. □

3.4. **The lifting theorem.** We now prove Beukers' lifting theorem.

Theorem 3.4 (Beukers, 2006)**.** *Under the assumptions of the Siegel–Shidlovskii Theorem, every homogeneous polynomial $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_m]$ such that*

$$P(f_1(\alpha), \ldots, f_m(\alpha)) = 0$$

can be lifted to a polynomial $Q \in \overline{\mathbb{Q}}[z, X_1, \ldots, X_m]$, homogeneous in $X_1, \ldots, X_m$, such that

$$Q(z, f_1(z), \ldots, f_m(z)) = 0 \quad \textit{and} \quad Q(\alpha, X_1, \ldots, X_m) = P(X_1, \ldots, X_m).$$

Proof. We continue with the notation of Sections 3.1 and 3.2. In particular, we assume that $\alpha = 1$. It suffices to prove that

$$\mathfrak{I}(1) = \mathrm{ev}_1(\mathfrak{I}). \tag{3.10}$$

By definition, $\mathrm{ev}_1(\mathfrak{I}) \subset \mathfrak{I}(1)$. Assume, for a contradiction, that $\mathrm{ev}_1(\mathfrak{I}) \subsetneq \mathfrak{I}(1)$.

By (A2), reproved in Appendix I, the ideal $\mathrm{ev}_1(\mathfrak{I})$ is prime. The ideal $\mathfrak{I}(1)$ is also prime, since it is the homogeneous ideal of algebraic relations among $f_1(1), \ldots, f_m(1)$. Therefore, if h and h_0 denote their respective heights (see [48, p. 193] for a definition), then $h_0 > h$. Recall that the basis of $\mathfrak{I}_\delta$ was chosen so that its specialization at $z = 1$ remains linearly independent. In particular, $\dim_{\mathbb{K}} \mathrm{ev}_1(\mathfrak{I})_\delta = \dim_{\mathbb{K}(z)} \mathfrak{I}_\delta$.

By the Hilbert–Serre Theorem, for sufficiently large δ, the numbers

$$\mu = \nu - \dim_{\mathbb{K}(z)} \mathfrak{I}_\delta = \nu - \dim_{\mathbb{K}} \mathrm{ev}_1(\mathfrak{I})_\delta, \quad \text{and} \quad \kappa := \nu - \dim_{\mathbb{K}} \mathfrak{I}(1)_\delta.$$

agree with polynomials of respective degrees $m - h - 1$ and $m - h_0 - 1$. Since $h_0 > h$, we may choose δ sufficiently large so that $\mu > 2[\mathbb{K}:\mathbb{Q}]\kappa$.

We now argue as in the proof of Theorem 3.3. Choose a complement of $\mathrm{ev}_1(\mathfrak{I})_\delta$ in $\mathfrak{I}(1)_\delta$. Its dimension is $\eta := \mu - \kappa$, and a basis of this complement provides η additional putative forms. These forms may be substituted for η of the Padé

forms occurring in the fixed preparation matrix, leaving only κ Padé forms. Exactly the same comatrix argument as in the proof of Theorem 3.3, together with $\mu > 2[\mathbb{K} : \mathbb{Q}]\kappa$, shows that the resulting matrix would be a forbidden matrix. This contradiction proves (3.10), and the theorem follows. □

3.5. **Algebraic independence measures.** We now prove Theorem 1.1 stated in the introduction.

We continue with the notation of Sections 3.1 and 3.2 and, in particular, we assume $\alpha = 1$. Assume for a contradiction that the conclusion of Theorem 1.1 fails. Then, for arbitrarily large real numbers γ there exist an integer d and infinitely many polynomials $P \in \mathbb{Z}[X_1, \dots, X_m]$ with degree at most d and height at most H such that

$$0 < |P(f_1(1), \dots, f_m(1))| < H^{-\gamma d^t}, \tag{3.11}$$

where $t := \operatorname{tr.deg}_{\mathbb{K}(z)}\big(f_1(z), \dots, f_m(z)\big)$. Fix a parameter γ large enough, an integer d associated, and a polynomial P satisfying (3.11), with H large enough, to be specified later. By adjoining the constant function 1 and homogenizing P, we may assume that $f_m = 1$ and that P is homogeneous. The fact that $P(f_1(1), \dots, f_m(1)) \neq 0$ yields $P \notin \mathfrak{I}(1)$. Let $\mathfrak{P}$ denote the homogeneous ideal of $\mathbb{K}[\boldsymbol{X}]$ generated by P and $\mathfrak{I}(1)$.

3.5.1. *Preliminary dimension estimates.* Let τ be a positive integer, to be chosen later, and set $\delta = d\tau$. Let ν and μ be the integers associated with δ in Section 3.2. We set

$$\kappa := \nu - \dim_{\mathbb{K}} \mathfrak{P}_\delta \quad \text{and} \quad \eta := \dim_{\mathbb{K}} \mathfrak{P}_\delta - \dim_{\mathbb{K}(z)}(\mathfrak{I}_{\mathbb{K}(z)})_\delta = \mu - \kappa.$$

Lemma 3.5. *There exist a positive integer τ_1 and a positive real number λ, both independent of γ, d and P, such that, for every $\tau \geq \tau_1$,*

$$\mu > 2[\mathbb{K} : \mathbb{Q}]\kappa \quad \text{and} \quad \mu \leq \lambda \tau^t d^t.$$

Proof. The second estimate follows from the fact that, for all sufficiently large δ, the integer μ is given by a polynomial in $\delta = d\tau$ of degree t, as observed in the proof of Theorem 3.3.

Set $\mathcal{A} := \mathbb{K}[\boldsymbol{X}]/\mathfrak{I}(1)$, and let $\mathcal{A}_\delta$ denote its homogeneous component of degree δ. Since $\mathfrak{I}(1)$ is prime, $\mathcal{A}$ is an integral domain. Let φ be its Hilbert polynomial. For all sufficiently large τ, uniformly with respect to $d \geq 1$, we have

$$\dim_{\mathbb{K}} \mathcal{A}_{\tau d} = \varphi(\tau d) \quad \text{and} \quad \dim_{\mathbb{K}} \mathcal{A}_{(\tau-1)d} = \varphi((\tau-1)d).$$

Let $\overline{P}$ denote the image of P in $\mathcal{A}$. Since $\overline{P} \neq 0$ and $\mathcal{A}$ is an integral domain, multiplication by $\overline{P}$ induces an isomorphism $\mathcal{A}_{(\tau-1)d} \longrightarrow (\overline{P}\mathcal{A})_{\tau d}$. Consequently, $\dim_{\mathbb{K}}(\overline{P}\mathcal{A})_{\tau d} = \varphi((\tau-1)d)$. Since

$$\frac{\varphi((\tau-1)d)}{\varphi(\tau d)} \longrightarrow 1$$

as τ tends to infinity, uniformly with respect to $d \geq 1$, there exists τ_1, independent of P, such that

$$\frac{\dim_{\mathbb{K}}(\overline{P}\mathcal{A})_\delta}{\dim_{\mathbb{K}} \mathcal{A}_\delta} > 1 - \frac{1}{2[\mathbb{K} : \mathbb{Q}]}$$

whenever $\tau \geq \tau_1$. Now $\kappa = \dim_{\mathbb{K}} \mathcal{A}_\delta - \dim_{\mathbb{K}}(\overline{P}\mathcal{A})_\delta$. Moreover, since $\operatorname{ev}_1(\mathfrak{I}_\delta) \subseteq \mathfrak{I}(1)_\delta$, we have $\dim_{\mathbb{K}} \mathcal{A}_\delta \leq \mu$. It follows that $2[\mathbb{K} : \mathbb{Q}]\kappa < \mu$, as desired. □

We henceforth fix $\tau = \tau_1$ and set $\varepsilon = \frac{1}{2}$. Writing $D := [\mathbb{K} : \mathbb{Q}]$, Lemma 3.5 gives

$$\mu - (1+\varepsilon)D\kappa > \frac{\mu}{4}. \tag{3.12}$$

We also assume, without loss of generality, that γ is chosen large enough so that

$$\gamma \geq 16D\lambda\tau^t. \tag{3.13}$$

3.5.2. *The putative forms.* Let $(M_n)_{n\geq n_1}$ be the sequence of preparation matrices constructed in Section 3.2 for $\delta = d\tau$ and $\varepsilon = 1/2$. Recall that its rows are the coefficient vectors of

$$L_1(\boldsymbol{Y}), \dots, L_\mu(\boldsymbol{Y}), \ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y}).$$

We set

$$\boldsymbol{\xi} := {}^{\mathrm{t}}(g_1(1), \dots, g_\nu(1)),$$

where $g_1, \dots, g_\nu$ are the monomials of degree δ in $f_1, \dots, f_m$. The quotient $\mathfrak{P}_\delta/\mathfrak{I}(1)_\delta$ identifies with the degree-δ component of the principal ideal generated by the image of P in $\mathcal{A}$. We may therefore choose monomials $A_1(\boldsymbol{X}), \dots, A_\eta(\boldsymbol{X})$ of degree $\delta - d$ such that the forms

$$N_i(\boldsymbol{Y}) := A_i(\boldsymbol{X})P(\boldsymbol{X}), \qquad 1 \leq i \leq \eta,$$

together with $\ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y})$ are linearly independent[1], since the latter belong to $\mathfrak{I}(1)_\delta$. These are our putative forms. In particular,

$$\boxed{N_i} \leq H, \qquad 1 \leq i \leq \eta.$$

3.5.3. *End of the proof of Theorem 1.1.* After reindexing the Padé forms, we may assume that

$$L_1(\boldsymbol{Y}), \dots, L_\kappa(\boldsymbol{Y}), N_1(\boldsymbol{Y}), \dots, N_\eta(\boldsymbol{Y}), \ell_1(1, \boldsymbol{Y}), \dots, \ell_{\nu-\mu}(1, \boldsymbol{Y}) \tag{3.14}$$

form a basis of the space of linear forms in $\boldsymbol{Y}$. Let $M_n' \in \mathcal{M}_\nu(\mathcal{O}_\mathbb{K})$ be the matrix whose rows are the coefficient vectors of the forms in (3.14). Then

$$M_n'\boldsymbol{\xi} = {}^{\mathrm{t}}\big(\boldsymbol{e}_n, \boldsymbol{e}_n', \boldsymbol{0}\big) =: \boldsymbol{\varepsilon}_n,$$

where $\boldsymbol{e}_n := \big(L_1(\boldsymbol{\xi}), \dots, L_\kappa(\boldsymbol{\xi})\big)$ and $\boldsymbol{e}_n' := \big(N_1(\boldsymbol{\xi}), \dots, N_\eta(\boldsymbol{\xi})\big)$. By Lemma 3.2, $\|\boldsymbol{e}_n\|_\infty \leq \rho_1 n^{-(\mu-1-\varepsilon)n}$. Moreover, since $N_i = A_iP$, we infer from (3.11) that there exists a positive real number θ, depending only on $f_1(1), \dots, f_m(1)$, such that

$$\|\boldsymbol{e}_n'\|_\infty \leq \theta^{\delta-d}H^{-\gamma d^t}. \tag{3.15}$$

Let C_n be the maximum of the absolute values of the entries in the first κ columns of $(M_n')^{-1}$, and let C_n' be the corresponding maximum over the following η columns. Up to factors depending only on δ, which are absorbed into the constants, we obtain

$$\|(M_n')^{-1}\boldsymbol{\varepsilon}_n\|_\infty \leq \rho_1 C_n n^{-(\mu-1-\varepsilon)n} + C_n' H^{-\gamma d^t}. \tag{3.16}$$

Set $H_n := \rho_2 n^{(1+\varepsilon)n}$. The first κ rows of M_n' have house at most H_n, the following η rows have house at most H, and the remaining rows have house

[1]Note that it follows from Beukers's lifting theorem that these forms span $\mathfrak{P}_\delta$, an observation that we shall not need here.

bounded independently of n and P. Since $\det M'_n$ is a non-zero algebraic integer, Liouville's inequality and the comatrix formula give

$$|\det M'_n|^{-1} \le \rho_3 H_n^{\kappa(D-1)} H^{\eta(D-1)}$$

for some positive constant ρ_3 independent of n and H. The cofactors corresponding to the first κ columns of $(M'_n)^{-1}$ involve at most $\kappa-1$ rows of house H_n and η rows of house H, whereas those corresponding to the following η columns involve at most κ rows of house H_n and $\eta-1$ rows of house H. Hence

$$\begin{aligned} C_n &\le \rho_4 n^{(1+\varepsilon)n(\kappa D-1)} H^{\eta D}, \\ C'_n &\le \rho_4 n^{(1+\varepsilon)n\kappa D} H^{\eta D-1}, \end{aligned} \tag{3.17}$$

where ρ_4 is independent of n and H. Combining (3.16) and (3.17), we obtain

$$\|(M'_n)^{-1}\boldsymbol{\varepsilon}_n\|_\infty \le \rho_1\rho_4 n^{-n(\mu-(1+\varepsilon)\kappa D)} H^{\eta D} + \rho_4 n^{(1+\varepsilon)n\kappa D} H^{\eta D-\gamma d^t-1}.$$

Let n be the unique integer such that

$$\rho_1^{-1} n^{n\mu} \le H^{\gamma d^t+1} < \rho_1^{-1}(n+1)^{(n+1)\mu}.$$

We may assume H is large enough to ensure that $n \ge n_1$. Therefore we have

$$\|(M'_n)^{-1}\boldsymbol{\varepsilon}_n\|_\infty \le 2\rho_1\rho_4 n^{-n(\mu-(1+\varepsilon)\kappa D)} H^{\eta D}.$$

Our choice of n, Lemma 3.5, (3.13), and the rough estimates $\gamma d^t+1 \le 2\gamma d^t$, and $(n+1)^{(n+1)} \le n^{2n}$, imply that

$$H \le \rho_5 n^{4n\mu\gamma^{-1}d^{-t}} \le \rho_5 n^{4n\lambda\tau^t\gamma^{-1}} \le \rho_5 n^{n/(4D)},$$

for some positive real number ρ_5. Therefore

$$\|(M'_n)^{-1}\boldsymbol{\varepsilon}_n\|_\infty \le \rho_6 n^{-n(\mu-(1+\varepsilon)\kappa D-\eta/4)}. \tag{3.18}$$

Using (3.12) and the fact that $\eta \le \mu$, we have $\mu-(1+\varepsilon)\kappa D-\eta/4 > 0$. Since n tends to infinity with H, if H is large enough, the quantity on the right-hand side of (3.18) is smaller than $\|\boldsymbol{\xi}\|_\infty$, which contradicts that $(M'_n)^{-1}\boldsymbol{\varepsilon}_n = \boldsymbol{\xi}$. Thus, M'_n is a forbidden matrix, which ends the proof. □

3.5.4. *Remark on effectivity.* Since Lang's work [25], several authors have investigated the effectivity of the constants appearing in algebraic independence measures for values of E-functions. When $f_1,\dots,f_m$ are algebraically independent, it follows from [23, Thm. 1] that one may take

$$C_1 = \exp\left(-\exp\left(C_3 d^{2m}\log(d+1)\right)\right) \text{ and } C_2 = \sqrt{m}\,4^m[\mathbb{K}:\mathbb{Q}]^{m+1}, \tag{3.19}$$

where $C_3 > 0$ depends only on $f_1,\dots,f_m$ and α. Obtaining an effective value for C_3 remained a long-standing problem, since it depends on the Shidlovskii constant arising in Shidlovskii's lemma; see, for instance, [17, Lem. 5.2, p. 221]. This problem was eventually solved through the work of Bertrand and Beukers [9] and of Bertrand, Chirskii, and Yebbou [10].

The constants occurring in the more general setting of Theorem 1.1 can likewise be bounded effectively. We omit the details, since making the constants explicit is standard, although rather cumbersome. Let us only indicate the additional ingredients that arise in the present situation.

The constants λ and $\max_i\left|\ell_i(1,\boldsymbol{Y})\right|$ do not appear in the classical argument, where $f_1,\dots,f_m$ are assumed to be algebraically independent. Moreover, in that case, the parameter τ_1 depends only on m and $[\mathbb{K}:\mathbb{Q}]$, and this dependence can

be made explicit. In our more general setting, these quantities also depend on the ideal of functional relations $\mathfrak{I}$. The main difficulty is therefore to bound the degrees and the heights of a generating set for $\mathfrak{I}$; see the discussion in [4].

Once these constants and the parameter n_1 arising from Shidlovskii's lemma have been bounded the remaining constants can be estimated by means of the results of [32].

We finally record, without proof, that the constants in Theorem 1.1 may be chosen in the form

$$C_1 = \exp\big(-\exp\big(C_3[\mathbb{K}:\mathbb{Q}]^{2t}d^{2t}\log\big([\mathbb{K}:\mathbb{Q}]d+1\big)\big)\big) \quad \text{and} \quad C_2 = C_4[\mathbb{K}:\mathbb{Q}]^{t+1},$$

where $C_3, C_4 > 0$ depend only on $f_1, \dots, f_m$ and α, but not on d, $[\mathbb{K}:\mathbb{Q}]$, or t.

4. The Siegel–Shidlovskii method at singular points

Theorems 3.3, 3.4, and 1.1 concern values of E-functions at a point that is regular for a differential system satisfied by these functions. This regularity assumption allows the derivatives of the initial Padé approximant to be rewritten as linear combinations of the original functions, thereby ensuring the existence of a sequence of preparation matrices.

At a singular point, such a rewriting is no longer available in general. Nevertheless, the successive derivatives of the Padé approximant can still be interpreted as linear forms in a larger family of functions. This observation extends the Siegel–Shidlovskii method to points that may a priori be singular. We do not pursue this framework in full generality here, but show that it is already sufficient to give a new proof of André's theorem (A1), which we restate below in the following slightly different form.

Theorem 4.1. *Every E-function satisfies a linear differential equation whose singularities belong to $\{0, \infty\}$.*

We then combine Theorem 4.1 with Theorem 1.1 to prove Corollary 1.2. We conclude this section by discussing some further consequences of Theorem 4.1. In Appendix II, we will use Theorem 4.1 to obtain a refinement of (A1), as well as a consequence concerning the minimal differential equation of a G-function.

4.1. The rational case.

We begin by proving a result for E-functions with rational coefficients, from which the general case will follow.

Theorem 4.2. *Let $f \in \mathbb{Q}[[z]]$ be an E-function. Then f satisfies a linear differential equation having 1 as an ordinary point.*

4.1.1. *Preliminary setting.* If $f = 0$, there is nothing to prove, so we assume that $f \neq 0$. Let i_0 be the least non-negative integer such that $f^{(i_0)}(1) \neq 0$, and let $\mu \geq 1$ be the order of the minimal differential equation satisfied by f. Let $\varepsilon > 0$ be a sufficiently small real number satisfying $\varepsilon\mu < 1$. We consider the infinite family of E-functions $\boldsymbol{f} := (f^{(i)}(z))_{i\geq 0} \subset \mathbb{Q}[[z]]$ and the corresponding family of indeterminates $\boldsymbol{X} := (X_i)_{i\geq 0}$.

Since we work over $\mathbb{Q}$, it is enough to consider linear forms in f and its derivatives. However, we do not fix in advance the number ν of derivatives involved, and allow it to depend on n. Consequently, the size of the preparation matrix M_n will also depend on n. For fixed n, the rows of M_n are divided into two blocks with, respectively, μ and $\nu - \mu$ rows:

- the first block consists of the specializations at $z = 1$ of certain derivatives of a Padé approximant involving $f, f', \dots, f^{(\mu-1)}$;
- the second block consists of the specializations at $z = 1$ of differential relations involving $f, f', \dots, f^{(\nu-1)}$.

4.1.2. *Forms arising from the differential equations.* For every $i \geq \mu$, we let d_i denote the minimum of the degree of $a_i(z)$ among all differential relations

$$a_0(z)f(z) + a_1(z)f'(z) + \cdots + a_i(z)f^{(i)}(z) = 0, \qquad a_0, \dots, a_i \in \mathbb{Z}[z],\ a_i \neq 0.$$

Since differentiating a relation of order i produces a relation of order $i+1$ with the same leading coefficient, the sequence $(d_i)_{i\geq\mu}$ is non-increasing. Let i_1 be such that this sequence is constant for $i \geq i_1$.

For each $i \geq \mu$, we choose a differential relation

$$a_{i,0}(z)f(z) + a_{i,1}(z)f'(z) + \cdots + a_{i,i}(z)f^{(i)}(z) = 0, \qquad a_{i,j} \in \mathbb{Z}[z],$$

with $\deg a_{i,i} = d_i$ by proceeding as follows. When $i \leq i_1$, we select an arbitrary differential relation for which $\deg a_{i,i} = d_i$. When $i > i_1$, we choose the differential relation obtained by differentiating $i - i_1$ times the relation of order i_1.

Set

$$\ell_i(z, \boldsymbol{X}) := a_{i,0}(z)X_0 + \cdots + a_{i,i}(z)X_i.$$

Thus $\ell_i(z, \boldsymbol{f}) = 0$, and, for any $i > i_1$,

$$\ell_i = \partial^{i-i_1}\ell_{i_1}$$

where ∂ is the derivation of $\mathbb{Z}[z, \boldsymbol{X}]$ defined by

$$\partial X_i := X_{i+1} \qquad \text{and} \qquad \partial a(z) := a'(z). \tag{4.1}$$

We claim that the infinite family of linear forms $\ell_i(1, \boldsymbol{X})$, $i \geq \mu$, is linearly independent. Suppose, to the contrary, that there exist integers $\lambda_1, \dots, \lambda_k$, with $\lambda_k \neq 0$, and indices $\mu \leq j_1 < \cdots < j_k$ such that $\lambda_1\ell_{j_1}(1, \boldsymbol{X}) + \cdots + \lambda_k\ell_{j_k}(1, \boldsymbol{X}) = 0$. The coefficients of

$$\lambda_1\ell_{j_1}(z, \boldsymbol{X}) + \cdots + \lambda_k\ell_{j_k}(z, \boldsymbol{X})$$

are then divisible by $z - 1$. Dividing by $z - 1$ yields a linear differential relation of order j_k with polynomial coefficients satisfied by f, whose leading coefficient is $\frac{\lambda_k a_{j_k,j_k}(z)}{z-1}$. This contradicts the minimality of the degree of a_{j_k,j_k}.

The houses of these linear forms may grow with i, but at most polynomially.

Lemma 4.3. *There exist a positive constant c_0 and a non-negative integer δ, independent of i, such that*

$$\overline{|\ell_i(1, \boldsymbol{X})|} \leq c_0 i^\delta$$

for every $i \geq \mu$.

Proof. Enlarging c_0 if necessary, we only need to establish the bound when $i \geq i_1$. Fix an integer $i \geq i_1$. Write

$$\ell_{i_1}(z, \boldsymbol{X}) := a_0(z)X_0 + \cdots + a_{i_1}(z)X_{i_1}, \qquad a_0, \dots, a_{i_1} \in \mathbb{Z}[z],$$

and set $\delta := \max_{0\leq s\leq i_1} \deg a_s$. By the Leibniz rule,

$$\partial^{i-i_1}(a_sX_s) = \sum_{j=0}^{\delta} \binom{i-i_1}{j} a_s^{(j)}(z)X_{s+i-i_1-j},$$

where the terms with $j > i - i_1$ are understood to be zero. For convenience, we extend the family $a_0, \dots, a_{i_1}$ by setting $a_s := 0$ for every $s \notin \{0, \dots, i_1\}$. Then the coefficient of X_{i-k} in $\ell_i = \partial^{i-i_1} \ell_{i_1}$ is

$$\sum_{j=0}^{\delta} \binom{i-i_1}{j} a_{i_1+j-k}^{(j)}(z). \tag{4.2}$$

Set

$$H_0 := \max_{\substack{0 \le s \le i_1 \\ 0 \le j \le \delta}} \left| a_s^{(j)}(1) \right|.$$

The absolute value of every coefficient of $\ell_i(1, \boldsymbol{X})$ is at most

$$H_0 \sum_{j=0}^{\delta} \binom{i-i_1}{j} \le c_0 i^{\delta}$$

for a suitable positive constant c_0. □

As in the proof of Lemma 4.3, we write

$$\ell_{i_1}(z, \boldsymbol{X}) := a_0(z) X_0 + \cdots + a_{i_1}(z) X_{i_1}.$$

Set

$$\eta := \min_{0 \le k \le i_1} \left\{ k + \operatorname{ord}_{z=1} a_{i_1-k}(z) \right\}. \tag{4.3}$$

The polynomials $a_0, \dots, a_{i_1}$ are relatively prime, since otherwise their common divisor could be removed, contradicting the minimality of $\deg a_{i_1,i_1}$. In particular, at least one of them does not vanish at 1, and hence $\eta \le i_1$.

Recall the convention that $a_s = 0$ for $s \notin \{0, \dots, i_1\}$. For every $i \ge i_1$, Formula (4.2) gives

$$\ell_i(z, \boldsymbol{X}) = \sum_{k=0}^{i} \sum_{j=0}^{k} \binom{i-i_1}{j} a_{i_1+j-k}^{(j)}(z) X_{i-k}.$$

By the definition of η, we have $a_{i_1+j-k}^{(j)}(1) = 0$ for every $k < \eta$ and every $j \le k$. Therefore $\ell_i(1, \boldsymbol{X})$ is a linear form in $X_0, \dots, X_{i-\eta}$. The coefficient of $X_{i-\eta}$ is

$$\sum_{j=0}^{\eta} \binom{i-i_1}{j} a_{i_1+j-\eta}^{(j)}(1). \tag{4.4}$$

Choose k realizing the minimum in (4.3), and set $j := \operatorname{ord}_{z=1} a_{i_1-k}(z)$. Then $\eta = k + j$ and $a_{i_1+j-\eta}^{(j)}(1) = a_{i_1-k}^{(j)}(1) \ne 0$. Consequently, not all the coefficients $a_{i_1+j-\eta}^{(j)}(1)$, $0 \le j \le \eta$, vanish. Since the polynomials $\binom{i-i_1}{j}$, $0 \le j \le \eta$, are linearly independent, Expression (4.4) is a non-zero polynomial in i. There thus exists an integer $i_2 \ge i_1$ such that the coefficient of $X_{i-\eta}$ in $\ell_i(1, \boldsymbol{X})$ is non-zero for every $i \ge i_2$. Enlarging i_2 if necessary, we may assume that $i_2 \ge i_0$.

4.1.3. *Forms arising from Padé approximants.* By the minimality of the differential equation satisfied by f, the functions $f, f', \dots, f^{(\mu-1)}$ are linearly independent over $\mathbb{Q}(z)$. Moreover, they form a solution of the companion system associated with this equation.

Recall that $\varepsilon > 0$ was chosen so that $\varepsilon\mu < 1$. We apply Proposition 3.1 to these μ functions with

$$\varepsilon_0 := \frac{\varepsilon}{6(\mu+1)}.$$

Let $n \geq n_1$, where n_1 is given by that proposition. There exist a positive real number ρ, independent of n, and polynomials $P_1, \dots, P_\mu \in \mathbb{Z}[z]$ of degree at most n such that:

- $\boxed{P_i} \leq \rho n^{(1+\varepsilon_0)n} \qquad (1 \leq i \leq \mu)$;
- setting $R(z) := P_1(z)f(z) + \cdots + P_\mu(z)f^{(\mu-1)}(z)$, we have
$$\operatorname{val} R(z) \geq \mu(n+1) - \varepsilon_0 n - 1;$$
- the functions $R, R', \dots, R^{(\mu-1)}$ are linearly independent over $\mathbb{Q}(z)$.

We now define a sequence $(R_k)_{k\geq 0}$ of linear forms in $\boldsymbol{X} = (X_i)_{i\geq 0}$ by

$$R_0(z, \boldsymbol{X}) := P_1(z)X_0 + \cdots + P_\mu(z)X_{\mu-1}$$

and

$$R_k(z, \boldsymbol{X}) := \partial^k R_0(z, \boldsymbol{X}),$$

where ∂ denotes the derivation of $\mathbb{Z}[z, \boldsymbol{X}]$ defined by (4.1). Then $R_k(z, \boldsymbol{f}) = R^{(k)}(z)$ for every $k \geq 0$, and R_k involves only the variables $X_0, \dots, X_{\mu+k-1}$.

4.1.4. *The preparation matrix.* We first prove the following statement.

Lemma 4.4. *There exist a positive real number c_1, independent of n and ε, two integers $\nu \geq i_2$ and $c_2 \geq 0$ satisfying $c_2 + \mu \leq \nu \leq \varepsilon_0 n + c_1$, positive real numbers ρ_1, ρ_2, and μ forms $L_1(\boldsymbol{X}), \dots, L_\mu(\boldsymbol{X})$ among $R_0(1, \boldsymbol{X}), \dots, R_{\mu+c_2-1}(1, \boldsymbol{X})$ such that:*

(a) $|L_i(\boldsymbol{f}(1))| \leq \rho_1 n^{-(\mu-1-\varepsilon)n} \qquad (1 \leq i \leq \mu)$;

(b) $\boxed{L_i} \leq \rho_2 n^{(1+\varepsilon)n} \qquad (1 \leq i \leq \mu)$;

(c) *the forms $L_1(\boldsymbol{X}), \dots, L_\mu(\boldsymbol{X}), \ell_\mu(1, \boldsymbol{X}), \dots, \ell_{\nu-1}(1, \boldsymbol{X})$ span the $\mathbb{Q}$-vector space of linear forms in $X_0, \dots, X_{\nu-\eta-1}$.*

Proof. Assertions (a) and (b) follow directly from the construction of the forms $R_k(z, \boldsymbol{X})$. To prove (c), we first show that there exist ν and c_2 such that $X_0, \dots, X_{\nu-\eta-1}$ belong to the $\mathbb{Q}$-vector space spanned by

$$R_0(1, \boldsymbol{X}), \dots, R_{\mu+c_2-1}(1, \boldsymbol{X}), \ell_\mu(1, \boldsymbol{X}), \dots, \ell_{\nu-1}(1, \boldsymbol{X}).$$

Since ν depends on n, the argument differs slightly from that used in the proof of Lemma 3.2. Set $i_3 := \max\{i_2 + 1, 2\mu\}$, and let

$$\Gamma(z) := (\gamma_{i,j}(z))_{0\leq i,j\leq i_3-1}$$

be defined by

$$\gamma_{i,j}(z) := \begin{cases} \text{the coefficient of } X_j \text{ in } R_i(z, \boldsymbol{X}), & 0 \leq i < \mu, \\ \text{the coefficient of } X_j \text{ in } \ell_i(z, \boldsymbol{X}), & \mu \leq i < i_3. \end{cases}$$

Arguing as in the proof of Lemma 3.2, we obtain that $\Gamma(z)$ is non-singular and that $\mathrm{ord}_{z=1}\Delta(z) \le \varepsilon_0 n + c_1$, where $\Delta(z) := \det\Gamma(z)$, for some positive real number c_1 independent of n and ε. Let $c_2 := \mathrm{ord}_{z=1}\Delta(z)$ and $\nu := i_3 + c_2$; Then $c_2 + \mu \le \nu$, and, after enlarging c_1 if necessary, $\nu \le \varepsilon_0 n + c_1$.

Set

$$\mathbf{L}(z, \boldsymbol{X}) := {}^{\mathrm{t}}\big(R_0(z, \boldsymbol{X}), \dots, R_{\mu-1}(z, \boldsymbol{X}), \ell_\mu(z, \boldsymbol{X}), \dots, \ell_{i_3-1}(z, \boldsymbol{X})\big).$$

By construction,

$$\mathbf{L}(z, \boldsymbol{X}) = \Gamma(z)\ {}^{\mathrm{t}}(X_0, \dots, X_{i_3-1}).$$

Hence, by the comatrix formula,

$$\Delta(z)\,{}^{\mathrm{t}}(X_0, \dots, X_{i_3-1}) = {}^{\mathrm{t}}\mathrm{Com}(\Gamma(z))\,\mathbf{L}(z, \boldsymbol{X}).$$

Recall that $\partial R_k = R_{k+1}$, that $\partial \ell_i$ is a $\mathbb{Q}[z]$-linear combination of $\ell_\mu, \dots, \ell_{i+1}$ for $i < i_1$, and that $\partial \ell_i = \ell_{i+1}$ for $i \ge i_1$. Applying ∂^{c_2} to the preceding identity, we obtain, for every $j \in \{0, \dots, i_3 - 1\}$,

$$\Delta^{(c_2)}(z)X_j + \sum_{h=0}^{c_2-1} \binom{c_2}{h} \Delta^{(h)}(z) X_{j+c_2-h} = \sum_{i=0}^{\mu+c_2-1} q_{i,j}(z) R_i(z, \boldsymbol{X}) + \sum_{i=\mu}^{\nu-1} r_{i,j}(z) \ell_i(z, \boldsymbol{X}),$$

for suitable polynomials $q_{i,j}, r_{i,j} \in \mathbb{Q}[z]$. Specializing at $z = 1$ and using that $\Delta^{(h)}(1) = 0$, for $h < c_2$, and $\Delta^{(c_2)}(1) \ne 0$, we obtain

$$X_j = \sum_{i=0}^{\mu+c_2-1} \lambda_{i,j} R_i(1, \boldsymbol{X}) + \sum_{i=\mu}^{\nu-1} \gamma_{i,j} \ell_i(1, \boldsymbol{X}), \qquad 0 \le j < i_3,$$

for suitable $\lambda_{i,j}, \gamma_{i,j} \in \mathbb{Q}$. Thus $X_0, \dots, X_{i_3-1}$ belong to the $\mathbb{Q}$-vector space spanned by

$$R_0(1, \boldsymbol{X}), \dots, R_{\mu+c_2-1}(1, \boldsymbol{X}), \ell_\mu(1, \boldsymbol{X}), \dots, \ell_{\nu-1}(1, \boldsymbol{X}). \tag{4.5}$$

We now prove by induction on j that X_j belongs to this space for every $j \le \nu - \eta - 1$. This has already been established for $j < i_3$. Let $i_3 \le j \le \nu - \eta - 1$ and assume that the result holds for all smaller indices. Since $j + \eta \ge i_2$, the coefficient λ of X_j in $\ell_{j+\eta}(1, \boldsymbol{X})$ is non-zero. Moreover, $\ell_{j+\eta}(1, \boldsymbol{X})$ is supported on $X_0, \dots, X_j$, and therefore

$$\ell_{j+\eta}(1, \boldsymbol{X}) = \lambda X_j + \sum_{k=0}^{j-1} \lambda_k X_k$$

for suitable $\lambda_k \in \mathbb{Q}$. Since $j + \eta \le \nu - 1$, the induction hypothesis shows that X_j belongs to the space spanned by (4.5). Consequently, the forms in (4.5) span the space of linear forms in $X_0, \dots, X_{\nu-\eta-1}$. Since the forms $\ell_\mu(1, \boldsymbol{X}), \dots, \ell_{\nu-1}(1, \boldsymbol{X})$ are linearly independent, their span has dimension $\nu - \mu$. The quotient by this span therefore has dimension $\mu - \eta$. We may thus select μ forms

$$L_1(\boldsymbol{X}), \dots, L_\mu(\boldsymbol{X})$$

among $R_0(1, \boldsymbol{X}), \dots, R_{\mu+c_2-1}(1, \boldsymbol{X})$ whose images span this quotient. These forms satisfy (c), while (a) and (b) follow from their construction. □

Let

$$M_n := (a_{n;i,j})_{1\le i,j\le \nu}$$

be the matrix defined by

$$a_{n;i,j} := \begin{cases} \text{the coefficient of } X_{j-1} \text{ in } L_i(\boldsymbol{X}), & 1 \le i \le \mu, \\ \text{the coefficient of } X_{j-1} \text{ in } \ell_{i-1}(1,\boldsymbol{X}), & \mu < i \le \nu. \end{cases}$$

Then $(M_n)_{n\ge n_1}$ is our sequence of preparation matrices. Unlike those considered in Section 3.2, the matrices M_n may not be invertible, as occurs when $\eta \ge 1$.

4.1.5. *The putative forms.* We now assume, for contradiction, that every differential equation satisfied by f has a singularity at 1. In particular, $a_{i_1}(1) = 0$, and hence $\eta \ge 1$.

Let $\kappa := \mu - \eta$. Let c_3 be a positive real number, independent of ε, such that

$$c_0(\varepsilon n + c_1 + \eta - 1)^\delta \le (\varepsilon n)^{c_3} \tag{4.6}$$

for every $n \ge 2n_1 \ge 2\varepsilon^{-1}$. Recall that i_0 is the least non-negative integer such that $f^{(i_0)}(1) \ne 0$. Since $\varepsilon\kappa \le \varepsilon\mu < 1$ and $\eta \ge 1$, we have $\eta - \varepsilon\kappa > 0$. We now choose $\varepsilon > 0$ sufficiently small that $\eta - \varepsilon\kappa - (c_3+1)\varepsilon > 0$. Using $k! = \mathcal{O}(k^{k+1/2})$, we may then choose an integer $n > 2n_1$ such that

$$(\varepsilon n + c_1 - 1)!\kappa\rho_1\rho_2^{\kappa-1} n^{-(\eta-\varepsilon\kappa)n}(\varepsilon n)^{c_3(\varepsilon n + c_1 - \kappa)} < \left|f^{(i_0)}(1)\right|. \tag{4.7}$$

We fix such an integer n and consider the preparation matrix associated with it.

The forms

$$\ell_{\nu+i}(1,\boldsymbol{X}), \quad 0 \le i \le \eta - 1,$$

are supported on the variables $X_0, \dots, X_{\nu-1}$. They constitute our set of putative forms.

4.1.6. *End of the proof of Theorem 4.2.* Arguing as at the end of the proof of Lemma 4.4, we obtain that $X_{\nu-\eta}, \dots, X_{\nu-1}$ belong to the $\mathbb{Q}$-vector space spanned by

$$L_1(\boldsymbol{X}), \dots, L_\mu(\boldsymbol{X}), \ell_\mu(1,\boldsymbol{X}), \dots, \ell_{\nu+\eta-1}(1,\boldsymbol{X}).$$

Consequently, after reindexing the forms L_i if necessary, the ν forms

$$L_1(\boldsymbol{X}), \dots, L_\kappa(\boldsymbol{X}), \ell_\mu(1,\boldsymbol{X}), \dots, \ell_{\nu+\eta-1}(1,\boldsymbol{X})$$

form a basis of the space of linear forms in $X_0, \dots, X_{\nu-1}$, where we recall that $\kappa = \mu - \eta$.

Let M_n' denote the coefficient matrix of these forms with respect to the basis $X_0, \dots, X_{\nu-1}$. It is invertible and it will be our forbidden matrix. Set

$$\boldsymbol{\xi} := {}^{\mathrm{t}}\big(f(1), f'(1), \dots, f^{(\nu-1)}(1)\big).$$

Since $\ell_i(1, \boldsymbol{f}(1)) = 0$, for every $i \ge \mu$, we have

$$M_n'\boldsymbol{\xi} = {}^{\mathrm{t}}\big(e_{n,1}, \dots, e_{n,\kappa}, 0, \dots, 0\big) =: \boldsymbol{\varepsilon}_n, \tag{4.8}$$

and $\|\boldsymbol{\varepsilon}_n\| \le \rho_1 n^{-(\mu-1-\varepsilon)n}$, by Lemma 4.4.

Let C_n denote the maximum of the absolute values of the entries in the first κ columns of $(M_n')^{-1}$. Arguing as in the proof of Theorem 3.3, we obtain

$$\|(M_n')^{-1}(\boldsymbol{\varepsilon}_n)\|_\infty \le \kappa C_n \rho_1 n^{-(\mu-1-\varepsilon)n}. \tag{4.9}$$

By (4.6), Lemma 4.3, and the fact that $\nu \le \varepsilon_0 n + c_1 \le \varepsilon n + c_1$, we have

$$\left|\ell_i(1, \boldsymbol{X})\right| \le (\varepsilon n)^{c_3} \qquad (i \le \nu + \eta - 1).$$

Moreover, Lemma 4.4 gives $\left|L_i\right| \le \rho_2 n^{(1+\varepsilon)n}$. Since $\det M_n'$ is a non-zero integer, we have $|\det M_n'| \ge 1$. The comatrix formula therefore yields

$$C_n \le (\nu-1)!\big(\rho_2 n^{(1+\varepsilon)n}\big)^{\kappa-1}(\varepsilon n)^{c_3(\nu-\kappa)}. \tag{4.10}$$

Combining (4.9) and (4.10), and using $\nu \le \varepsilon n + c_1$, we obtain

$$\begin{aligned}\|(M_n')^{-1}\boldsymbol{\varepsilon}_n\|_\infty &\le \kappa\rho_1 n^{-(\mu-1-\varepsilon)n}(\nu-1)!\big(\rho_2 n^{(1+\varepsilon)n}\big)^{\kappa-1}(\varepsilon n)^{c_3(\nu-\kappa)}\\ &\le (\varepsilon n + c_1 - 1)!\kappa\rho_1\rho_2^{\kappa-1} n^{-(\eta-\varepsilon\kappa)n}(\varepsilon n)^{c_3(\varepsilon n + c_1 - \kappa)}\\ &< \big|f^{(i_0)}(1)\big|,\end{aligned}$$

where the last inequality follows from (4.7). Since $i_0 \le \nu - 1$, the definition of $\|\boldsymbol{\xi}\|_\infty$ contradicts (4.8), showing that M_n' is a forbidden matrix. This ends the proof. □

4.2. **Proof of the general case.** We could prove Theorem 4.1 along the same lines as Theorem 4.2. However, more than one additional linear form would then be needed to obtain a contradiction, forcing us to consider monomials in f and its derivatives. Since this would lead to rather cumbersome notation, we prefer to derive it from Theorem 4.2, using elementary properties of differential operators.

We first recall the following classical lemma, for which we provide a proof for completeness.

Lemma 4.5. *Let $\alpha \in \overline{\mathbb{Q}}$ and let $L_1, \dots, L_h \in \overline{\mathbb{Q}}(z)\langle\partial\rangle$ be differential operators having α as an ordinary point. Then there exists a common left multiple of $L_1, \dots, L_h$ in $\overline{\mathbb{Q}}(z)\langle\partial\rangle$ having α as an ordinary point.*

Proof. It is enough to consider two operators L_1 and L_2. Let M denote their monic least common left multiple. In a common differential extension, the solution space of M is the sum of the solution spaces of L_1 and L_2. Indeed, if G denotes their greatest common right divisor, then

$$\operatorname{ord}_\partial M = \operatorname{ord}_\partial L_1 + \operatorname{ord}_\partial L_2 - \operatorname{ord}_\partial G,$$

while the intersection of the solution spaces of L_1 and L_2 is the solution space of G.

Since α is an ordinary point of L_1 and L_2, all their local solutions are holomorphic at α. Hence all local solutions of M are holomorphic at α. Thus α is either an ordinary point or an apparent singularity of M. By [1, Appendix, Prop. 7], whose proof applies verbatim over the algebraically closed field $\overline{\mathbb{Q}}$, the operator M admits a left multiple $\widetilde{M} \in \overline{\mathbb{Q}}(z)\langle\partial\rangle$ having α as an ordinary point. Since $\widetilde{M}$ is also a left multiple of L_1 and L_2, the result follows. The general case follows by induction. □

Proof of Theorem 4.1. We first establish the following claim: for every $\alpha \in \mathbb{C}^\times$, every E-function is annihilated by a differential operator in $\overline{\mathbb{Q}}(z)\langle\partial\rangle$ having α as an ordinary point.

Let f be an E-function. If $\alpha \notin \overline{\mathbb{Q}}$, then α is an ordinary point of every differential operator in $\overline{\mathbb{Q}}(z)\langle\partial\rangle$, since the finite singularities of such an operator are algebraic. The claim therefore holds in this case.

Suppose now that $\alpha \in \overline{\mathbb{Q}}^\times$. Replacing $f(z)$ by $f(\alpha z)$, it is enough to treat the case $\alpha = 1$. Since the coefficients of f belong to a number field, we may write

$$f = \beta_1 f_1 + \cdots + \beta_h f_h,$$

where $\beta_1, \dots, \beta_h \in \overline{\mathbb{Q}}$ and $f_1, \dots, f_h \in \mathbb{Q}[[z]]$ are E-functions (see [4, Lem. 19]). By Theorem 4.2, for every $i \in \{1, \dots, h\}$ there exists an operator $L_i \in \mathbb{Q}(z)\langle\partial\rangle$ such that $L_i(f_i) = 0$ and 1 is an ordinary point of L_i. By Lemma 4.5, the operators $L_1, \dots, L_h$ admit a common left multiple L in $\overline{\mathbb{Q}}(z)\langle\partial\rangle$ having 1 as an ordinary point. Since $L(f_i) = 0$ for every i, we obtain

$$L(f) = \beta_1 L(f_1) + \cdots + \beta_h L(f_h) = 0.$$

After undoing the change of variable $z \mapsto \alpha z$, this proves the claim.

Let now $M \in \overline{\mathbb{Q}}(z)\langle\partial\rangle$ be the monic minimal differential operator annihilating f. We claim that every non-zero finite singularity of M is apparent. Indeed, let $\alpha \in \mathbb{C}^\times$. By the first part of the proof, there exists an operator $L_\alpha \in \overline{\mathbb{Q}}(z)\langle\partial\rangle$ annihilating f and having α as an ordinary point. By the minimality of M, there exists $Q_\alpha \in \overline{\mathbb{Q}}(z)\langle\partial\rangle$ such that $L_\alpha = Q_\alpha M$. Consequently, every local solution of M is also a local solution of L_α, and is therefore holomorphic at α. Hence, if α is a singularity of M, it is an apparent singularity.

We may now apply the desingularization theorem [1, Appendix, Thm. 6] to M. Its proof applies verbatim over the algebraically closed field $\overline{\mathbb{Q}}$ and yields a monic left multiple $\widetilde{M} \in \overline{\mathbb{Q}}(z)\langle\partial\rangle$ of M such that every ordinary point and every apparent singularity of M is an ordinary point of $\widetilde{M}$. Since all the non-zero finite singularities of M are apparent, the operator $\widetilde{M}$ has no non-zero finite singularity. Finally, since $M(f) = 0$ and $\widetilde{M}$ is a left multiple of M, we have $\widetilde{M}(f) = 0$. This proves the theorem. □

Proof of André's theorem (A1). Let f be an E-function. As observed in the proof of Theorem 4.1, any point $\alpha \in \mathbb{C}^\times$ is either an ordinary point or an apparent singularity of the minimal differential operator annihilating f. Therefore, the only possible non-apparent singularities of this operator are 0 and ∞. □

4.3. **Proof of Corollary 1.2.** We are now ready to deduce Corollary 1.2 from Theorems 1.1 and 4.1.

Proof of Corollary 1.2. Every element of $\mathbf{E}$ can be written as the value at $z = 1$ of an E-function. For each $i \in \{1, \dots, r\}$, let f_i be an E-function such that $\xi_i = f_i(1)$. By Theorem 4.1, the function f_i satisfies a differential equation

$$a_{i,0}(z) f_i(z) + a_{i,1}(z) f_i'(z) + \cdots + a_{i,m_i}(z) f_i^{(m_i)}(z) = 0 \tag{4.11}$$

having no non-zero finite singularity. In particular, $a_{i,m_i}(1) \neq 0$. The column vector ${}^{\mathrm{t}}(f_i(z), f_i'(z), \dots, f_i^{(m_i-1)}(z))$ is a solution of the linear differential system

$Y'(z) = A_i(z)Y(z)$, where

$$A_i(z) := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\dfrac{a_{i,0}(z)}{a_{i,m_i}(z)} & \cdots & \cdots & \cdots & -\dfrac{a_{i,m_i-1}(z)}{a_{i,m_i}(z)} \end{pmatrix}. \tag{4.12}$$

The functions $f_i^{(k)}$, with $1 \le i \le r$ and $0 \le k \le m_i - 1$, are therefore the coordinates of a vector solution of the differential system $Y'(z) = A(z)Y(z)$, where $A(z)$ is the block-diagonal matrix

$$A(z) := A_1(z) \oplus \cdots \oplus A_r(z) = \begin{pmatrix} A_1(z) & & \\ & \ddots & \\ & & A_r(z) \end{pmatrix}.$$

The point 1 is regular with respect to this system. The result now follows directly from Theorem 1.1. Moreover, t is equal to the transcendence degree over $\mathbb{C}(z)$ of the functions $f_i^{(k)}$, with $1 \le i \le r$ and $0 \le k \le m_i - 1$, or equivalently, to the transcendence degree over $\mathbb{C}(z)$ of the differential field generated by $f_1, \dots, f_r$. □

4.3.1. *Remarks on Corollary 1.2.* Let us end this section with a few comments on Corollary 1.2.

Recall that a real transcendental number ξ is called a *U_d-number* if, for every $w \ge 1$, there exist non-zero polynomials $P \in \mathbb{Z}[X]$ of degree at most d and height at most $H \ge 2$ such that

$$0 < |P(\xi)| \le H^{-w}.$$

A real transcendental number is called a *U-number* if it is a U_d-number for some positive integer d. In particular, the U_1-numbers are exactly the Liouville numbers.

The question of whether some values of E-functions may be U-numbers, and in particular Liouville numbers, is a longstanding problem with roots in the pioneering work of Siegel [44]. It was fully resolved only recently by Fischler and Rivoal [20], who used a desingularization procedure due to Beukers together with a Galois-theoretic argument. Their proof relies on Beukers' lifting theorem and on André's result (A1). As a direct consequence of Corollary 1.2, we obtain a new proof of their result, one that does not invoke Beukers' lifting theorem.

Corollary 4.6. *No value of an E-function at an algebraic point is a U-number. In particular, no such value is a Liouville number.*

Proof. Let $\xi \in \mathbf{E}$. If ξ is not real or if it is algebraic, there is nothing to prove. Assume therefore that ξ is real and transcendental. Let $d \ge 1$ be fixed. By Corollary 1.2, there exist positive constants A and B, depending only on ξ and d, such that, for every non-zero polynomial $P \in \mathbb{Z}[X]$ of degree at most d and height at most H,

$$|P(\xi)| \ge AH^{-B}.$$

It follows immediately that ξ is not a U_d-number. Since d is arbitrary, ξ is not a U-number. □

A general form of Liouville's inequality may be stated as follows. Given algebraic numbers $\alpha_1, \dots, \alpha_m$ and a polynomial $P \in \mathbb{Z}[X_1, \dots, X_m]$ of (total) degree at most d height at most H, one has

$$\text{either } P(\alpha_1, \dots, \alpha_m) = 0, \qquad \text{or } |P(\alpha_1, \dots, \alpha_m)| \geq C_1 H^{-C_2},$$

where C_1 depends on d and $\alpha_1, \dots, \alpha_m$, whereas C_2 depends only on $\alpha_1, \dots, \alpha_m$; see, for instance, [47, p. 84–85]. In this respect, Corollary 1.2 may be viewed as a Liouville-type inequality for elements of $\mathbf{E}$. This estimate allows one to work with elements of $\mathbf{E}$ in much the same way as with algebraic numbers, though the bounds are far less sharp. In the case of values of the Bessel function, this was already observed by Siegel, just after stating his main result [44] (see also [46, pp. 29–30]).

Just as Liouville's inequality implies the transcendence of $\sum_{n\geq 0} 10^{-n!}$, Corollary 1.2, together with the effective estimates discussed in Section 3.5.4, implies that a number of the form

$$\sum_{n=0}^{\infty} e^{-u_n}$$

is transcendental over $\mathbf{E}$, provided that the strictly increasing sequence of positive integers $(u_n)_{n\geq 0}$ grows sufficiently rapidly. More precisely, it is enough to assume that, for every real number $A > 0$,

$$\lim_{n\to\infty} \frac{\log u_{n+1}}{u_n^A \log u_n} = +\infty.$$

For example, one may take $u_n := {}^n n$, where

$${}^n n := \underbrace{n^{n^{\cdot^{\cdot^{\cdot^{n}}}}}}_{n},$$

denotes the power tower of n copies of n.

Appendix I. Specialization of differential ideals

The main purpose of this appendix is to provide an elementary proof of André's result (A2) stated in the Introduction. It will be deduced from the more general Proposition I.2 below, which is itself a consequence of [8, Thm. 1.6.1(3)].

Recall that, for a field k, an ideal $\mathcal{I} \subset k[X_1, \dots, X_m]$ is said to be *absolutely prime* if, for every field extension K of k, the ideal generated by $\mathcal{I}$ in $K[X_1, \dots, X_m]$ remains prime. This is equivalent to the fact that the fraction field of $k[X_1, \dots, X_m]/\mathcal{I}$ is a regular extension of k (see, for instance, [48, Thm. 39]). Moreover, when k has characteristic zero, this is equivalent to the fact that k is algebraically closed in the fraction field of $k[X_1, \dots, X_m]/\mathcal{I}$.

Let $\overline{\mathbb{Q}}[z]_{(z-\alpha)}$ denote the localization of $\overline{\mathbb{Q}}[z]$ at the prime ideal $(z-\alpha)$, that is, the ring of rational functions that are regular at α. An element $P \in \overline{\mathbb{Q}}[z]_{(z-\alpha)}[X_1, \dots, X_m]$ can naturally be specialized at α by setting $\mathrm{ev}_\alpha(P) := P(\alpha, X_1, \dots, X_m) \in \overline{\mathbb{Q}}[X_1, \dots, X_m]$. For an ideal $\mathcal{I} \subset \overline{\mathbb{Q}}(z)[X_1, \dots, X_m]$, we set

$$\mathrm{ev}_\alpha(\mathcal{I}) := \mathrm{ev}_\alpha\big(\mathcal{I} \cap \overline{\mathbb{Q}}[z]_{(z-\alpha)}[X_1, \dots, X_m]\big).$$

Remark I.1. We have $\mathrm{ev}_\alpha(\mathcal{I}) = \mathrm{ev}_\alpha\big(\mathcal{I} \cap \overline{\mathbb{Q}}[z][X_1, \dots, X_m]\big)$. Indeed, if $P = \frac{Q}{d(z)} \in \overline{\mathbb{Q}}[z]_{(z-\alpha)}[X_1, \dots, X_m]$, where $Q \in \overline{\mathbb{Q}}[z][X_1, \dots, X_m]$ and $d(\alpha) \neq 0$, then $\mathrm{ev}_\alpha(P) = \mathrm{ev}_\alpha\Big(\frac{Q}{d(\alpha)}\Big)$.

Proposition I.2. *Let D be a derivation on $\overline{\mathbb{Q}}(z)[\boldsymbol{X}] = \overline{\mathbb{Q}}(z)[X_1, \dots, X_m]$ acting on $\overline{\mathbb{Q}}(z)$ as the usual derivation and on $\boldsymbol{X} := {}^{\mathrm{t}}(X_1, \dots, X_m)$ by*

$$D\boldsymbol{X} = A(z)\boldsymbol{X} \tag{I.1}$$

for some matrix $A(z) \in \mathcal{M}_m(\overline{\mathbb{Q}}(z))$. Let $\mathcal{I} \subset \overline{\mathbb{Q}}(z)[\boldsymbol{X}]$ be a D-invariant absolutely prime ideal, and let $\alpha \in \overline{\mathbb{Q}}$ be a point that is not a pole of $A(z)$. Then $\mathrm{ev}_\alpha(\mathcal{I})$ is prime.

By a classical result of Krull [24], the specialization of an absolutely prime ideal $\mathcal{I}$ of $\overline{\mathbb{Q}}(z)[X_1, \dots, X_m]$ remains prime for all but finitely many α. Proposition I.2 strengthens this by showing that, when $\mathcal{I}$ is D-invariant, this finite exceptional set is actually contained in the set of singular points of the differential system (I.1).

Before proving Proposition I.2, we show how it implies André's result (A2). Let $\mathfrak{I}$ be the ideal appearing in (A2), and let $\mathfrak{I}_{\overline{\mathbb{Q}}(z)} := \mathfrak{I}\,\overline{\mathbb{Q}}(z)[X_1, \dots, X_m]$ be its extension to $\overline{\mathbb{Q}}(z)[X_1, \dots, X_m]$. This ideal is invariant under the derivation D defined by $D\boldsymbol{X} = A(z)\boldsymbol{X}$. Moreover, the field $\overline{\mathbb{Q}}(z)(f_1, \dots, f_m)$ is a regular extension of $\overline{\mathbb{Q}}(z)$. Indeed, since E-functions are entire, $\overline{\mathbb{Q}}(z)(f_1, \dots, f_m)$ is a field of single-valued meromorphic functions on $\mathbb{C}$. Hence $\overline{\mathbb{Q}}(z)$ is algebraically closed in $\overline{\mathbb{Q}}(z)(f_1, \dots, f_m)$ and the extension is regular, so that $\mathfrak{I}_{\overline{\mathbb{Q}}(z)}$ is absolutely prime. Proposition I.2 now shows that $\mathrm{ev}_\alpha(\mathfrak{I}_{\overline{\mathbb{Q}}(z)})$ is prime. Since $\mathrm{ev}_\alpha(\mathfrak{I}_{\overline{\mathbb{Q}}(z)}) = \mathrm{ev}_\alpha(\mathfrak{I})$, the latter ideal is prime.

I.1. **Proof of Proposition I.2.** We now turn to the proof of Proposition I.2. By replacing z with $z + \alpha$, it suffices to prove the result for $\alpha = 0$. The proof proceeds in two steps. We first establish the result when the derivation is the trivial extension of $\frac{d}{dz}$. We then reduce the general case to this special case by using a gauge transformation associated with a fundamental matrix of solutions. Since this matrix has entries in $\overline{\mathbb{Q}}[[z]]$, the corresponding change of variables is naturally carried out over the Laurent series field $\overline{\mathbb{Q}}((z))$.

We identify $\overline{\mathbb{Q}}[[z]][\boldsymbol{X}]$ with a subring of $\overline{\mathbb{Q}}[\boldsymbol{X}][[z]]$ in a natural way. Thus every element $P \in \overline{\mathbb{Q}}[[z]][\boldsymbol{X}]$ admits a unique expansion

$$P = \sum_{n \geq 0} P_n(\boldsymbol{X}) z^n, \qquad P_n(\boldsymbol{X}) \in \overline{\mathbb{Q}}[\boldsymbol{X}],$$

and we define the specialization morphism

$$\mathrm{ev}_0 : \overline{\mathbb{Q}}[[z]][\boldsymbol{X}] \longrightarrow \overline{\mathbb{Q}}[\boldsymbol{X}]$$

by $\mathrm{ev}_0(P) := P_0(\boldsymbol{X})$. For an ideal $\mathcal{I} \subset \overline{\mathbb{Q}}((z))[\boldsymbol{X}]$, we set

$$\mathrm{ev}_0(\mathcal{I}) := \mathrm{ev}_0\big(\mathcal{I} \cap \overline{\mathbb{Q}}[[z]][\boldsymbol{X}]\big).$$

We extend the derivation $\frac{d}{dz}$ from $\overline{\mathbb{Q}}((z))$ to $\overline{\mathbb{Q}}((z))[\boldsymbol{X}]$ by defining

$$\frac{d}{dz} X_i = 0, \quad 1 \leq i \leq m.$$

Lemma I.3. *Let $\mathcal{I} \subset \overline{\mathbb{Q}}((z))[\boldsymbol{X}]$ be a $\frac{d}{dz}$-invariant ideal. Then*

$$\operatorname{ev}_0(\mathcal{I}) = \mathcal{I} \cap \overline{\mathbb{Q}}[\boldsymbol{X}]. \tag{I.2}$$

In particular, if $\mathcal{I}$ is prime, then so is $\operatorname{ev}_0(\mathcal{I})$.

Proof. The final assertion follows immediately from (I.2), since the contraction of a prime ideal is prime. It remains to prove the equality.

The inclusion $\mathcal{I} \cap \overline{\mathbb{Q}}[\boldsymbol{X}] \subseteq \operatorname{ev}_0(\mathcal{I})$ is immediate. Conversely, let $Q(\boldsymbol{X}) \in \operatorname{ev}_0(\mathcal{I})$. By definition, there exists $R(z, \boldsymbol{X}) \in \mathcal{I} \cap \overline{\mathbb{Q}}[[z]][\boldsymbol{X}]$ such that $Q(\boldsymbol{X}) = \operatorname{ev}_0(R)$. Let d be the total degree of R in $\boldsymbol{X}$, and consider a $\overline{\mathbb{Q}}((z))$-basis

$$P_1(z, \boldsymbol{X}), \dots, P_r(z, \boldsymbol{X})$$

of the vector space of elements of $\mathcal{I}$ having total degree at most d in $\boldsymbol{X}$. After multiplying the P_i by suitable powers of z, we may assume that $P_1, \dots, P_r \in \overline{\mathbb{Q}}[[z]][\boldsymbol{X}]$.

The proof of [12, Lem. 3.1] carries over *mutatis mutandis* to linear forms with coefficients in $\overline{\mathbb{Q}}[[z]]$. As a consequence, we can choose such a basis so that $\operatorname{ev}_0(P_1), \dots, \operatorname{ev}_0(P_r)$ are linearly independent over $\overline{\mathbb{Q}}$.

Since R has total degree at most d in $\boldsymbol{X}$, there exist $g_1(z), \dots, g_r(z) \in \overline{\mathbb{Q}}((z))$ such that

$$R(z, \boldsymbol{X}) = g_1(z)P_1(z, \boldsymbol{X}) + \cdots + g_r(z)P_r(z, \boldsymbol{X}).$$

The linear independence of $\operatorname{ev}_0(P_1), \dots, \operatorname{ev}_0(P_r)$ implies that all the g_i belong to $\overline{\mathbb{Q}}[[z]]$. Indeed, otherwise, multiplying this equality by the smallest power of z that removes all the poles and then applying ev_0 would yield a non-trivial linear relation among $\operatorname{ev}_0(P_1), \dots, \operatorname{ev}_0(P_r)$. Thus

$$Q(\boldsymbol{X}) = \sum_{i=1}^{r} g_i(0) \operatorname{ev}_0(P_i). \tag{I.3}$$

Let $\boldsymbol{P}(z, \boldsymbol{X})$ denote the column vector with entries $P_1(z, \boldsymbol{X}), \dots, P_r(z, \boldsymbol{X})$. Since $\mathcal{I}$ is invariant under $\frac{d}{dz}$ and this derivation does not increase the degree in $\boldsymbol{X}$, there exist matrices $B_0(z) = \mathrm{I}_r, B_1(z), B_2(z), \dots$ with entries in $\overline{\mathbb{Q}}((z))$ such that

$$\boldsymbol{P}^{(k)}(z, \boldsymbol{X}) = B_k(z)\boldsymbol{P}(z, \boldsymbol{X}), \qquad k \geq 0.$$

The linear independence of the entries of $\operatorname{ev}_0(\boldsymbol{P})$ implies that every $B_k(z)$ has entries in $\overline{\mathbb{Q}}[[z]]$. Expanding the entries of $\boldsymbol{P}$ into their formal Taylor series, we obtain

$$\boldsymbol{P}(z, \boldsymbol{X}) = \sum_{k=0}^{\infty} \frac{z^k}{k!} \boldsymbol{P}^{(k)}(0, \boldsymbol{X}) = \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} B_k(0) \right) \operatorname{ev}_0(\boldsymbol{P}).$$

The matrix

$$C(z) := \sum_{k=0}^{\infty} \frac{z^k}{k!} B_k(0)$$

has entries in $\overline{\mathbb{Q}}[[z]]$ and satisfies $C(0) = \mathrm{I}_r$. Hence $C(z) \in \mathrm{GL}_r(\overline{\mathbb{Q}}[[z]])$ and

$$\operatorname{ev}_0(\boldsymbol{P}) = C(z)^{-1} \boldsymbol{P}(z, \boldsymbol{X}).$$

It follows that $\operatorname{ev}_0(P_1), \dots, \operatorname{ev}_0(P_r) \in \mathcal{I}$. Equation (I.3) therefore implies that $Q \in \mathcal{I}$. Thus $\operatorname{ev}_0(\mathcal{I}) \subseteq \mathcal{I} \cap \overline{\mathbb{Q}}[\boldsymbol{X}]$, which completes the proof. □

We now show that the general case reduces to Lemma I.3. The derivation D extends to $\overline{\mathbb{Q}}((z))[\boldsymbol{X}]$ by setting $Dg := g'(z)$ for every $g \in \overline{\mathbb{Q}}((z))$, and $D\boldsymbol{X} := A(z)\boldsymbol{X}$. Consider the differential system $Y'(z) = A(z)Y(z)$. Since $A(z)$ has no pole at 0, there exists a unique *fundamental matrix of solutions* $F(z) \in \mathrm{GL}_m(\overline{\mathbb{Q}}[[z]])$ such that

$$F'(z) = A(z)F(z) \qquad \text{and} \qquad F(0) = \mathrm{I}_m.$$

Define

$$\begin{aligned} \phi_F : \overline{\mathbb{Q}}((z))[\boldsymbol{X}] &\longrightarrow \overline{\mathbb{Q}}((z))[\boldsymbol{X}], \\ P(z, \boldsymbol{X}) &\longmapsto P(z, F(z)\boldsymbol{X}). \end{aligned}$$

This is an automorphism, with inverse $\phi_{F^{-1}}$, and it restricts to an automorphism of $\overline{\mathbb{Q}}[[z]][\boldsymbol{X}]$.

Lemma I.4. *Let $\mathcal{I} \subset \overline{\mathbb{Q}}((z))[\boldsymbol{X}]$ be a D-invariant ideal. Then $\phi_F(\mathcal{I})$ is $\frac{d}{dz}$-invariant.*

Proof. We claim that, for every $P \in \overline{\mathbb{Q}}((z))[\boldsymbol{X}]$,

$$\phi_F(DP) = \frac{d}{dz}\phi_F(P).$$

The two sides agree on $\overline{\mathbb{Q}}((z))$ and satisfy the same Leibniz rule relative to ϕ_F. It is therefore enough to check the identity on $X_1, \dots, X_m$. In vector notation, we have

$$\phi_F(D\boldsymbol{X}) = \phi_F(A(z)\boldsymbol{X}) = A(z)F(z)\boldsymbol{X} = F'(z)\boldsymbol{X} = \frac{d}{dz}(F(z)\boldsymbol{X}) = \frac{d}{dz}\phi_F(\boldsymbol{X}).$$

Now let $P \in \phi_F(\mathcal{I})$. Then $P = \phi_F(Q)$ for some $Q \in \mathcal{I}$, and therefore

$$\frac{dP}{dz} = \frac{d}{dz}\phi_F(Q) = \phi_F(DQ) \in \phi_F(\mathcal{I}),$$

since $DQ \in \mathcal{I}$. □

We may now prove Proposition I.2.

Proof of Proposition I.2. By assumption, $\mathcal{I}$ is absolutely prime. Hence the extended ideal $\widehat{\mathcal{I}} := \mathcal{I}\overline{\mathbb{Q}}((z))[\boldsymbol{X}]$ is prime. Since ϕ_F is an automorphism, the ideal $\phi_F(\widehat{\mathcal{I}}) \subset \overline{\mathbb{Q}}((z))[\boldsymbol{X}]$ is also prime. By Lemma I.4, it is $\frac{d}{dz}$-invariant. By Lemma I.3, the specialization $\mathrm{ev}_0(\phi_F(\widehat{\mathcal{I}}))$ is therefore prime.

Now ϕ_F and $\phi_{F^{-1}}$ both preserve $\overline{\mathbb{Q}}[[z]][\boldsymbol{X}]$. Moreover, since $F(0) = \mathrm{I}_m$, we have $\mathrm{ev}_0(\phi_F(P)) = \mathrm{ev}_0(P)$ for every $P \in \overline{\mathbb{Q}}[[z]][\boldsymbol{X}]$. Consequently, $\mathrm{ev}_0(\phi_F(\widehat{\mathcal{I}})) = \mathrm{ev}_0(\widehat{\mathcal{I}})$, and hence $\mathrm{ev}_0(\widehat{\mathcal{I}})$ is prime.

Finally, since $\widehat{\mathcal{I}}$ is the extension of $\mathcal{I}$ to $\overline{\mathbb{Q}}((z))[\boldsymbol{X}]$, it follows that $\mathrm{ev}_0(\mathcal{I}) = \mathrm{ev}_0(\widehat{\mathcal{I}})$. Hence $\mathrm{ev}_0(\mathcal{I})$ is prime, and the proof is complete. □

Appendix II. Further consequences of Theorem 4.1

Although (A1) is the only part of André's result that has been used so far in the transcendence theory of E-functions, André actually proved in [6] a more precise description of the singularities of the minimal differential equation of a non-polynomial E-function: it is regular singular at 0 and irregular at infinity, with slopes in $\{0, 1\}$, and it admits a full basis of solutions in the class of Nilsson–Gevrey series of arithmetic type and order -1. Obtaining this more precise result

requires the full strength of the theory of G-operators, including results of the Chudnovskys, Katz, and André himself. Here and in Theorem II.1 below, the slopes refer to those of the local Newton polygon at infinity of a differential equation as considered in [30].

Our approach does not seem to recover this full result. Nevertheless, we show that Theorem 4.1 easily implies the following refinement of (A1). In particular, the Siegel–Shidlovskii method, as revisited in the present work, also shows that the minimal differential equation satisfied by any G-function is regular singular at the origin.

Theorem II.1. *The minimal differential equation satisfied by a non-polynomial E-function has no non-trivial singularities outside $\{0, \infty\}$. Moreover,*

- *it is regular singular at 0;*
- *it is irregular at infinity, with slopes belonging to the interval $[0, 1]$.*

Furthermore, the minimal differential equation satisfied by any G-function is regular singular at 0.

Let us first recall some facts about differential operators. Set $\partial = d/dz$. We denote by $\mathcal{F}$ the Fourier–Laplace transform, that is, the automorphism of the Weyl algebra $\mathbb{C}[z]\langle\partial\rangle$ defined by $\mathcal{F}(z) := -\partial$ and $\mathcal{F}(\partial) := z$. It has order four, and its inverse, denoted by $\overline{\mathcal{F}} := \mathcal{F}^{-1}$, satisfies $\overline{\mathcal{F}}(z) = \partial$ and $\overline{\mathcal{F}}(\partial) = -z$.

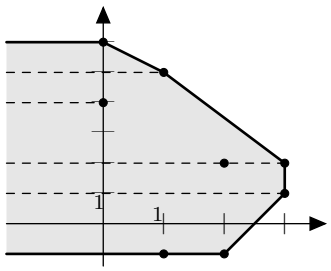
In the proof of Theorem II.1, it will be more convenient to use the Newton–Ramis polygon instead of the more classical local Newton polygons. The Newton–Ramis polygon $\mathcal{N}(L)$ of a differential operator

$$L := a_m(z)\partial^m + a_{m-1}(z)\partial^{m-1} + \cdots + a_0(z)$$

is defined as the convex hull of the horizontal half-lines whose right-hand endpoints are

$$(i, \mathrm{val}(a_i) - i) \qquad \text{and} \qquad (i, \deg(a_i) - i),$$

for those i such that $a_i \neq 0$ (see an example below). This convention differs from the one used for local Newton polygons of differential equations in [30]: the positive slopes of $\mathcal{N}(L)$ are the finite positive slopes of the local Newton polygon at 0, whereas the opposites of the negative slopes of $\mathcal{N}(L)$ are the finite positive slopes of the local Newton polygon at infinity, computed after the change of variable $z \mapsto z^{-1}$; see [31, Rem. p. 179].



The Newton-Ramis polygon of the differential operator $(3z^5 - z^4)\partial^3 + (z^4 + z)\partial^2 + (z^6 + z^2 + 3z + 1)\partial + (z^6 + z^4)$.

We shall use the following properties of the Newton–Ramis polygon.

(a) If $L = L_1 \cdots L_k$, then the set of finite slopes of $\mathcal{N}(L)$ is the union of the sets of finite slopes of the polygons $\mathcal{N}(L_i)$; see [30, Lem. 3.45].
(b) The operator L is regular singular at 0, respectively at infinity, if and only if $\mathcal{N}(L)$ has no positive, respectively no negative, slope; see [30, Ex. 3.47].
(c) Under the Fourier–Laplace transform, a slope t is mapped to $-\frac{t}{t+1}$; see [6, Sec. 5]. This transformation is understood as an involution of $\mathbb{R} \cup \{\infty\}$, exchanging -1 and ∞.
(d) A change of variable $z \mapsto z^\nu$, with $\nu \in \mathbb{Z} \setminus \{0\}$, multiplies the slopes of the Newton–Ramis polygon by ν; see [31, p. 180]. In particular, the change of variable $z \mapsto z^{-1}$ changes the signs of the slopes.
(e) Let L be the least common left multiple of $L_1, \dots, L_k$. Then the set of finite slopes of $\mathcal{N}(L)$ is the union of the sets of finite slopes of the polygons $\mathcal{N}(L_i)$.

    This statement is not explicitly formulated in [30], but it follows from the formal decomposition theorem. Indeed, in a common formal differential extension and after a finite ramification, the solution space of an operator M decomposes according to its exponential parts: its solutions are finite sums of terms of the form $f e(q)$, where $f \in \mathbb{C}\big(\!\big(z^{1/\star}\big)\!\big)\big[(z^a)_{a\in\mathbb{C}}, \log z\big]$ and $e(q)$ satisfies $z\frac{d}{dz}e(q) = qe(q)$, with $q \in \bigcup_{r\geq 1} z^{-1/r}\mathbb{C}[z^{-1/r}]$; see [30, p. 76]. The non-negative slopes of $\mathcal{N}(M)$ are the opposites of the valuations of the exponential parts q; see [30, Thm. 3.54 and Rem. 3.55]. In such a common formal extension, the solution space of a least common left multiple is the sum of the solution spaces of the operators involved. Its exponential parts, and hence its non-negative slopes, are therefore exactly the union of theirs. The corresponding assertion for the negative slopes follows by applying the change of variable $z \mapsto z^{-1}$.
(f) Let $f(z) := \sum_{n\geq 0} \frac{a_n}{n!} z^n$, $g(z) := \sum_{n\geq 0} a_n z^n$, and $h(z) := z^{-1} g(z^{-1})$. If L annihilates f and ν is sufficiently large, then $\partial^\nu \mathcal{F}(L)$ annihilates h. Conversely, if L annihilates h, then $\overline{\mathcal{F}}(L)$ annihilates f; see [6, Sec. 2].

We shall also use the following elementary observation, for which we provide a proof for completeness.

Lemma II.2. *Let u be a non-zero holonomic power series and let q be a positive integer. If L is the minimal operator annihilating $u(z)$, then the pullback of L under the change of variable $z \mapsto z^q$ is the minimal operator annihilating $u(z^q)$. In particular, the slopes of the minimal operator of $u(z^q)$ are obtained from those of L by multiplication by q.*

Proof. Set $\theta := z\partial$. The order of the minimal differential operator annihilating u is the dimension over $\mathbb{C}(z)$ of the vector space spanned by $u, \theta u, \theta^2 u, \dots$. Moreover, $\theta^j\big(u(z^q)\big) = q^j(\theta^j u)(z^q)$ for every $j \geq 0$. We use the standard fact that $\mathbb{C}(z^{1/q})$ and $\mathbb{C}((z))$ are linearly disjoint over $\mathbb{C}(z)$. Hence elements of $\mathbb{C}((z))$ which are linearly independent over $\mathbb{C}(z)$ remain linearly independent over $\mathbb{C}(z^{1/q})$. Let $x := z^q$. A linear relation over $\mathbb{C}(z)$ among $u(z^q), \theta u(z^q), \dots, \theta^r u(z^q)$ becomes a linear relation over $\mathbb{C}(x^{1/q})$ among $u(x), \theta u(x), \dots, \theta^r u(x)$. It follows that the minimal operators of $u(z)$ and $u(z^q)$ have the same order. The pullback of L under $z \mapsto z^q$ annihilates $u(z^q)$ and has the same order as L. It is therefore minimal. The assertion concerning the slopes follows from (d). □

Our proof is divided into five steps.

Step 1. *Every E-function is annihilated by an operator whose Newton–Ramis polygon has no infinite slope.*

By definition, the Newton–Ramis polygon of an operator has an infinite slope if and only if $\mathrm{val}(a_m) < \deg(a_m)$. Theorem 4.1 implies that every E-function is annihilated by a differential operator with polynomial coefficients whose only possible singularities are 0 and infinity. Its leading coefficient is therefore of the form cz^ν, for some $c \neq 0$ and some non-negative integer ν. Hence

$$\mathrm{val}(a_m) = \deg(a_m) = \nu,$$

so the associated Newton–Ramis polygon has no infinite slope.

Step 2. *The Newton–Ramis polygon of the minimal operator annihilating a G-function has no slope equal to* 1.

Let $g(z) := \sum_{n\geq 0} a_n z^n$ and $f(z) := \sum_{n\geq 0} \frac{a_n}{n!} z^n$ be the associated G- and E-functions. Let L be an operator annihilating f as in Step 1. Since the map $t \longmapsto -\frac{t}{t+1}$ sends ∞ to -1, the Newton–Ramis polygon of $\mathcal{F}(L)$ has no slope equal to -1. Choose ν such that $\Psi := \partial^\nu \mathcal{F}(L)$ annihilates $z^{-1}g(z^{-1})$, as in (f). Since ∂ has no finite non-zero slope, property (a) shows that $\mathcal{N}(\Psi)$ still has no slope equal to -1. Let Φ be the operator obtained from Ψ by the change of variable $z \mapsto z^{-1}$. Then Φ annihilates $zg(z)$ and, by (d), $\mathcal{N}(\Phi)$ has no slope equal to 1. Consequently, the operator Φz, where z denotes right multiplication by z, annihilates g. Since the operator of multiplication by z has no finite non-zero slope, property (a) shows that $\mathcal{N}(\Phi z)$ has no slope equal to 1. The minimal operator of g is a right factor of every operator annihilating g. Property (a) therefore implies that its Newton–Ramis polygon has no slope equal to 1.

Step 3. *The Newton–Ramis polygon of the minimal operator annihilating a G-function has no positive integral slope.*

We argue by contradiction. Let g be a G-function, let L be its minimal operator, and suppose that $\mathcal{N}(L)$ has an integral slope $t \geq 1$. Write $g(z) := \sum_{i=0}^{t-1} z^i g_i(z^t)$. For every i, the series $z^i g_i(z^t)$ is the Hadamard product of $g(z)$ and $z^i/(1-z^t)$, and is therefore a G-function. Moreover, each g_i is a G-function, since the class of G-functions is stable under sections of their coefficient sequences. Let L_i be the minimal operator annihilating $z^i g_i(z^t)$, and let M be the least common left multiple of $L_0, \dots, L_{t-1}$. Then M annihilates g, so the minimal operator L is a right factor of M. By (a), the slope t of $\mathcal{N}(L)$ is therefore a slope of $\mathcal{N}(M)$. By (e), it must be a slope of $\mathcal{N}(L_i)$ for at least one i. Multiplication of a solution by the monomial z^i amounts to a regular singular gauge transformation and does not change the slopes. Hence the minimal operator of $g_i(z^t)$ also has a slope equal to t. By Lemma II.2, the minimal operator of $g_i(z)$ has a slope equal to 1. This contradicts Step 2.

Step 4. *The minimal operator annihilating a G-function is regular singular at* 0.

By (b), it is enough to prove that its Newton–Ramis polygon has no positive slope. Suppose, on the contrary, that the minimal operator L of a G-function g has a positive slope p/q, where p and q are positive integers. The series $g(z^q)$ is

again a G-function. By Lemma II.2, the Newton–Ramis polygon of its minimal operator has the positive integral slope p. This contradicts Step 3. Thus $\mathcal{N}(L)$ has no positive slope, and L is regular singular at 0.

Step 5. *The non-zero slopes of the Newton–Ramis polygon of the minimal operator of an E-function belong to $[-1,0)$. In particular, this operator is regular singular at 0.*

Let f be an E-function and let g be its associated G-function. Let L denote the minimal operator of the G-function $zg(z)$. Let Ψ be the operator obtained from L by the change of variable $z \mapsto z^{-1}$. Thus, Ψ annihilates $z^{-1}g(z^{-1})$. By (f), the operator $\Phi := \overline{\mathcal{F}}(\Psi)$ annihilates f. Let $t \neq 0$ be a finite slope of $\mathcal{N}(\Phi)$. By (c), $-\frac{t}{t+1}$ is a slope of $\mathcal{N}(\Psi)$. After the change of variable $z \mapsto z^{-1}$, property (d) shows that $\frac{t}{t+1}$ is a slope of $\mathcal{N}(L)$. By Step 4, the minimal operator L has no positive slope. Hence $\frac{t}{t+1} \leq 0$, unless $t = -1$, in which case the quotient is infinite. It follows that $-1 \leq t < 0$. The minimal operator of f is a right factor of Φ. Property (a) therefore shows that all of its finite non-zero slopes also belong to $[-1,0)$. Moreover, by Step 1, the minimal operator of f is a right factor of an operator having no infinite slope, and hence it has no infinite slope either. In particular, it has no positive slope and is regular singular at 0.

We are now ready to conclude the proof of Theorem II.1. The finite non-zero slopes of the local Newton polygon at infinity are the opposites of the negative Newton–Ramis slopes. Hence, Step 5 implies that the finite slopes of the local Newton polygon at infinity of the minimal operator of an E-function belong to $[0,1]$, as claimed. It only remains to check that ∞ cannot be regular singular when f is non-polynomial. An entire solution of a differential equation that is regular singular at infinity has at most polynomial growth at infinity. By Cauchy's estimates, such an entire function is a polynomial. Thus the minimal differential equation of a non-polynomial E-function is irregular at infinity. This completes the proof of Theorem II.1. □

Declaration on the use of AI. All new mathematical ideas and results presented in this article are entirely due to the authors. AI tools were used solely to check spelling and identify typographical errors in mathematical expressions.

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