COMPUTING BASIS OF SOLUTIONS OF ANY MAHLER EQUATION

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In the paper [FP25b] we provide an algorithm to compute a fundamental matrix of solution to any linear Mahler system $\phi_p(Y) = AY$, where $A \in \mathrm{GL}_m(\mathbb{K}((z)))$, ϕ_p denotes the map $z \mapsto z^p$ and $\mathbb{K}$ is any field. Our main contribution is the computation of a pair (P, Θ) such that

$$\phi_p(P)\Theta = AP,$$

P is a non-singular matrix whose entries are Puiseux series with coefficients in $\mathbb{K}$ and $\Theta \in \mathrm{GL}_m(\mathbb{K}[z^{-1}])$ is a block upper triangular matrix with constant diagonal blocks. Algorithm 1 [FP25b] returns the first coefficients in the Puiseux expansion of P and the matrix Θ for such a pair (P, Θ). When $\mathbb{K}$ is an effective field of characteristic 0 and $A(z)$ has entries in $\mathbb{K}(z)$, this algorithm has been implemented in Python by the authors: `https://github.com/CFaverjon/Mahler-Systems/blob/main/Algorithm_FundamentalMatrixSolutions_MahlerSystems.py`. Once the output of this algorithm is known, computing a fundamental matrix of solutions for the Mahler system $\phi_p(Y) = AY$ becomes straightforward, as explained in [FP25b].

In this note, we present the output of our implemented algorithm for various examples found in the literature.

Example 1. A system associated to the Rudin-Shapiro sequence (Sequence A020985 in the OEIS) is the 2-Mahler system with matrix

$$A(z) = \begin{pmatrix} 1 & z \\ 1 & -z \end{pmatrix}.$$

Our algorithm returns an admissible pair (P, Θ) with

$$P = \begin{pmatrix} 4z^2 & -4z^2 + 2z - 2 \\ -4z & -2z^2 + 6z - 2 \end{pmatrix} + \mathcal{O}(z^3), \quad \text{and} \quad \Theta = \begin{pmatrix} -2 & 4 - 1/z \\ 0 & 1 \end{pmatrix}.$$

Example 2. A system associated to the Baum-Sweet sequence (Sequence A086747 in the OEIS) is the 2-Mahler system with matrix

$$A(z) = \begin{pmatrix} 0 & 1 \\ 1 & -z \end{pmatrix}.$$

Our algorithm returns an admissible pair (P, Θ) with

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(z), \quad \text{and} \quad \Theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

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Example 3. In the paper [CDDM18], a running example is the 3-Mahler system with matrix

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{z^6(1+z)(1-z^{21}-z^{30})}{z^3(1-z^3+z^6)(1-z^7-z^{10})} & \frac{1-z^{28}-z^{31}-z^{37}-z^{40}}{z^3(1-z^3+z^6)(1-z^7-z^1)} \end{pmatrix}.$$

Our algorithm returns an admissible pair (P, Θ) with

$$P = \left(\begin{array}{c|c} \begin{array}{c} 3z^9 - 2z^8 + 2z^7 \\ -2z^6 + z^5 - z^4 + z^3 \end{array} & \begin{array}{c} z^{\frac{19}{2}} - z^{\frac{17}{2}} + z^{\frac{15}{2}} - z^{\frac{13}{2}} + z^{\frac{11}{2}} \\ -z^{\frac{9}{2}} + z^{\frac{7}{2}} - z^{\frac{5}{2}} + z^{\frac{3}{2}} - z^{\frac{1}{2}} + z^{-\frac{1}{2}} \end{array} \\ \hline z^9 & -z^{\frac{15}{2}} + z^{\frac{9}{2}} - z^{\frac{3}{2}} + z^{-\frac{3}{2}} \end{array}\right) + \mathcal{O}(z^{10})$$

and $\Theta = \mathrm{I}_2$.

Example 4. In characteristic 3, the system associated with the Carlitz's function f_1 is the 3-Mahler system

$$A(z) = \begin{pmatrix} \theta - z^3 & \theta - z^3 \\ 0 & 1 \end{pmatrix}.$$

Our algorithm returns an admissible pair (P, Θ) with

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(z), \quad \text{and} \quad \Theta = \begin{pmatrix} \theta & \theta \\ 0 & 1 \end{pmatrix}.$$

Example 5. The first example given in Section 8 of the paper [FP25a] is the 2-Mahler system with matrix

$$A(z) = \begin{pmatrix} 0 & 1 \\ -\frac{1+z}{z^8} & \frac{z^2+z^3+z^7}{z^8} \end{pmatrix}.$$

Our algorithm returns an admissible pair (P, Θ) with

$$P = \frac{1}{2}\left(\begin{array}{c|c} \begin{array}{c} -5z^{15} + 17z^{14} - 5z^{13} - 3z^{12} \\ +2z^{11} + 3z^{10} - 3z^9 - 3z^8 - 2z^7 \\ +8z^6 - 3z^5 - z^4 - z^3 \\ +3z^2 - 2z - z^{-1} + 2z^{-2} - z^{-3} \end{array} & \begin{array}{c} 7z^{15} - 29z^{14} + 9z^{13} + 7z^{12} - 4z^{11} \\ -7z^{10} + 5z^9 + 7z^8 + 4z^7 \\ -16z^6 + 7z^5 - z^4 + 5z^3 \\ -7z^2 + 4z - 2 + z^{-1} + z^{-3} \end{array} \\ \hline \begin{array}{c} -3z^{14} + 12z^{12} - 5z^{10} - 3z^6 \\ +5z^4 - 3z^2 + 1 - z^{-2} + z^{-4} - z^{-6} \end{array} & \begin{array}{c} 5z^{14} - 20z^{12} + 9z^{10} - 2z^8 + 7z^6 \\ -9z^4 + 5z^2 - 3 + z^{-2} + z^{-4} + z^{-6} \end{array} \end{array}\right) + \mathcal{O}(z^{16})$$

and

$$\Theta = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{pmatrix}.$$

Example 6. Let a_n be the nth integer whose 3-adic expansion contains no 1 (Sequence A005823 in the OEIS). The sequence (a_n) lists the numerators of the left endpoints of the triadic Cantor set. As a 2-regular sequence it has an associated 2-Mahler system, which is the system with matrix

$$A(z) = \begin{pmatrix} \frac{4z+1}{3z^2+6z+3} & \frac{z-2}{z^2+2z+1} \\ \frac{-z}{3z^2+6z+3} & \frac{3}{3z^2+6z+3} \end{pmatrix}.$$

Our algorithm returns an admissible pair (P, Θ) with

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathcal{O}(z), \quad \text{and} \quad \Theta = \begin{pmatrix} \frac{1}{3} & -2 \\ 0 & 1 \end{pmatrix}.$$

Example 7. Let a_n denote the number of distinct factors of length n in the Thue-Morse sequence. According to [Du93], the characteristic series of (a_n) is solution of the 2-Mahler equation

$$\begin{aligned}
&(2z^9 - 2z^8 + 2z^6 - 2z^4 + z^3 + z^2 - z)h(z)\\
&+ \left(-2z^{11} - 2z^{10} - 2z^9 + 4z^8 - 4z^7 - 4z^6 + 3z^5 + 3z^4 - 6z^3 + z^2 + z + 1\right) h(z^2)\\
&+ \left(4z^{13} - 4z^{12} + 8z^{11} - 4z^{10} + 4z^9 + 4z^7 - 4z^6 + 8z^5 - 5z^4 + 8z^3 - 6z^2 + 4z - 3\right) h(z^4)\\
&-(4z^{11} - 4z^{10} + 4z^9 - 2z^8 + 8z^7 - 8z^6 + 8z^5 - 4z^4 + 4z^3 - 4z^2 + 4z - 2)h(z^8) = 0.
\end{aligned}$$

The associated companion matrix admits an admissible pair (P, Θ) with

$$P = \begin{pmatrix} -\frac{3z^2}{2} - \frac{z}{2} + \frac{1}{2} & z^2 - 1 & \frac{3z^2}{2} + \frac{3z}{2} + \frac{3}{2} \\ \frac{1}{4} - \frac{3z^2}{4} & -1 & \frac{7z^2}{4} + \frac{7}{4} \\ \frac{1}{8} & -1 & \frac{15}{8} \end{pmatrix} + \mathcal{O}(z^3), \quad \text{and} \quad \Theta = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -\frac{1}{2} & 0 & \frac{3}{2} \end{pmatrix}.$$

Example 8. The sequence of "Euclidian matching" introduced in [Du93, Example 88] is related with the 2-Mahler system with matrix

$$A(z) = \begin{pmatrix} \frac{z^3-2z^2-z}{z^3+z^2-z-1} & \frac{z^3+z}{z^2+2z+1} & \frac{-z^4+z^3+z^2+z}{z^3+z^2-z-1} \\ \frac{z^3+z}{z^3+z^2-z-1} & \frac{-z^2+z}{z^2+2z+1} & \frac{-2z^2}{z^3+z^2-z-1} \\ \frac{-2z^2}{z^3+z^2-z-1} & \frac{2z}{z^2+2z+1} & \frac{z^3+z}{z^3+z^2-z-1} \end{pmatrix}.$$

Our algorithm returns an admissible pair (P, Θ) with

$$P = \begin{pmatrix} \frac{4z^3}{5} + \frac{2z^2}{5} & \frac{5z^{\frac{10}{3}}}{6} + \frac{z^{\frac{7}{3}}}{2} + \frac{z^{\frac{4}{3}}}{6} & \frac{5z^{\frac{11}{3}}}{6} + \frac{z^{\frac{8}{3}}}{2} + \frac{z^{\frac{5}{3}}}{6} \\ z^3 + \frac{3z^2}{5} + \frac{z}{5} & \frac{5z^{\frac{10}{3}}}{6} + \frac{z^{\frac{7}{3}}}{2} + \frac{z^{\frac{4}{3}}}{6} & z^{\frac{11}{3}} + \frac{2z^{\frac{8}{3}}}{3} + \frac{z^{\frac{5}{3}}}{3} \\ z^3 + \frac{3z^2}{5} + \frac{z}{5} & z^{\frac{10}{3}} + \frac{2z^{\frac{7}{3}}}{3} + \frac{z^{\frac{4}{3}}}{3} & z^{\frac{11}{3}} + \frac{2z^{\frac{8}{3}}}{3} + \frac{z^{\frac{5}{3}}}{3} \end{pmatrix} + \mathcal{O}(z^4),$$

and

$$\Theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Example 9. The sequence with general term $\mathsf{v}_5(n!)$ is 5-regular. It is associated with the 5-Mahler system whose matrix is

$$\frac{1}{a(z)} \begin{pmatrix} 26z^4 + 21z^3 + 16z^2 + 11z + 6 & -25z^4 - 19z^3 - 13z^2 - 7z - 1 & 4z^4 + 3z^3 + 2z^2 + z \\ 5z^4 + 5z^3 + 5z^2 + 5z + 5 & 0 & 0 \\ -75z^4 - 50z^3 - 25z^2 + 25 & 95z^4 + 65z^3 + 35z^2 + 5z - 25 & -15z^4 - 10z^3 - 5z^2 + 5 \end{pmatrix}$$

where $a(z) = 5z^8 + 10z^7 + 15z^6 + 20z^5 + 25z^4 + 20z^3 + 15z^2 + 10z + 5$. Our algorithm returns an admissible pair (P, Θ) with

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \mathcal{O}(z), \quad \text{and} \quad \Theta = \begin{pmatrix} \frac{6}{5} & -\frac{1}{5} & 0 \\ 1 & 0 & 0 \\ 5 & -5 & 1 \end{pmatrix}.$$

Example 10. The sequence whose general term in the sum of bits it the Gray code (see [AS92, Example 15]) representation of n is associated with the 2-Mahler system with matrix

$$A(z) = \begin{pmatrix} \frac{z^2+z+2}{z^3+z^2+z+1} & \frac{-1}{z^2+1} \\ \frac{2}{z^3+z^2+z+1} & \frac{z-1}{z^2+1} \end{pmatrix}.$$

Our algorithm returns an admissible pair (P, Θ) with

$$\begin{pmatrix} z+2 & z^2-1 \\ 2 & z^2+z-1 \end{pmatrix} + \mathcal{O}(z^3), \quad \text{and} \quad \Theta = \mathrm{I}_2 \,.$$

Let us explain on this example how one can compute further terms in the expansion of $P(z)$. Using the equality

$$P(z) = A(z)^{-1} P(z^2) \Theta = A(z)^{-1} P(z^2) \,,$$

that the valuation at $z = 0$ of $A^{-1}(z)$ is -1, and writing $P(z) = \sum_{n=0}^{\infty} P_n z^n$, we observe that P_n, for $n \geq 2$, corresponds to the coefficient of z^n in the expansion of

$$A(z)^{-1}(P_0 + z^2 P_1 + \cdots + z^{2\lceil \frac{n}{2} \rceil} P_{\lceil \frac{n}{2} \rceil}) \,.$$

Using this recursive formula and using the initial values P_0, P_1, P_2, we derive the following expansion for $P(z)$ up to z^{10}

$$\begin{pmatrix} -2z^{10} - z^9 + z^7 - z^5 + z^3 + z + 2 & 3z^{10} + 2z^9 + z^8 + z^6 + 2z^5 + z^4 + z^2 - 1 \\ -2z^{10} - 2z^9 - 2z^5 + 2 & 3z^{10} + 3z^9 + z^8 + z^7 + z^6 + 3z^5 + z^4 + z^3 + z^2 + z - 1 \end{pmatrix} .$$

Example 11. The Tower of Hanoi morphism on the alphabet $\{a, b, c, \overline{a}, \overline{b}, \overline{c}\}$ [AS03, § 6.4] is given by:

$$a \mapsto a\overline{c}, \quad b \mapsto c\overline{b}, \quad c \mapsto b\overline{a}, \quad \overline{a} \mapsto ac, \quad \overline{b} \mapsto cb, \quad \overline{c} \mapsto ba$$

It is associated with a 2-Mahler system whose matrix is

$$A(z) = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{z^3+1} & \frac{-z}{z^3+1} & \frac{z^2}{z^3+1} \\ 0 & 0 & 0 & \frac{-z}{z^3+1} & \frac{z^2}{z^3+1} & \frac{1}{z^3+1} \\ 0 & 0 & 0 & \frac{z^7}{z^8+z^5} & \frac{z^5}{z^8+z^5} & \frac{-z}{z^3+1} \\ 0 & 0 & \frac{1}{z} & \frac{1}{z^3+1} & \frac{-z}{z^3+1} & \frac{-1}{z^4+z} \\ 0 & \frac{1}{z} & 0 & \frac{-z}{z^3+1} & \frac{-1}{z^4+z} & \frac{z^5}{z^8+z^5} \\ \frac{1}{z} & 0 & 0 & \frac{-1}{z^4+z} & \frac{1}{z^3+1} & \frac{-z}{z^3+1} \end{pmatrix} .$$

Our algorithm returns an admissible pair (P, Θ) where P is equal, modulo z^6, to

$$\begin{pmatrix} z^5+z^2 & z^3+1 & z^4+z & z^3+1 & z^4+z & z^5+z^2 \\ z^4+z & z^5+z^2 & z^3+1 & z^5+z^2 & z^3+1 & z^4+z \\ z^3+1 & z^4+z & z^5+z^2 & z^4+z & z^5+z^2 & z^3+1 \\ z^5+z^2 & z^3+1 & z^4+z & z^3+1 & z^4+z & z^5+z^2+1/z \\ z^4+z & z^5+z^2 & z^3+1 & z^5+z^2+1/z & z^3+1 & z^4+z \\ z^3+1 & z^4+z & z^5+z^2 & z^4+z & z^5+z^2+1/z & z^3+1 \end{pmatrix}$$

and

$$\Theta = \begin{pmatrix} 0 & 0 & 1 & 1/z & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1/z \\ 1 & 0 & 0 & 0 & 1/z & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} .$$

Let us compute the matrices H and e_C in that case. We can transform Θ into an upper triangular matrix, which yields a new pair where P is replaced with PQ^{-1}, where

$$Q := \begin{pmatrix} 5-1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

and Θ becomes

$$\begin{pmatrix} -1 & 0 & 0 & -1/z & -1+\frac{1}{2z} & \frac{1}{2z} \\ 0 & 1 & 0 & 1 & -\frac{1}{2z} & \frac{1}{2z} \\ 0 & 0 & 1 & \frac{1}{z} & \frac{1}{2z} & 1+\frac{1}{2z} \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

Using Algorithm 2 in [FP25b], we obtain that

$$H = \begin{pmatrix} 1 & 0 & 0 & \boldsymbol{\xi}_1 & -\frac{1}{2}\boldsymbol{\xi}_{-1} & -\frac{1}{2}\boldsymbol{\xi}_1 \\ 0 & 1 & 0 & 0 & -\frac{1}{2}\boldsymbol{\xi}_1 & \frac{1}{2}\boldsymbol{\xi}_{-1} \\ 0 & 0 & 1 & \boldsymbol{\xi}_{-1} & \frac{1}{2}\boldsymbol{\xi}_1 & \frac{1}{2}\boldsymbol{\xi}_{-1} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\boldsymbol{\xi}_1 = \sum_{k\geq 1} z^{-\frac{1}{2^k}}$ and $\boldsymbol{\xi}_{-1} = \sum_{k\geq 1}(-1)^k z^{-\frac{1}{2^k}}$. Last, using Algorithm 3 in [FP25b], we obtain

$$e_C = \begin{pmatrix} e_{-1} & 0 & 0 & 0 & \frac{e_{-1}}{2} - 1/2 & 0 \\ 0 & 1 & 0 & \frac{1}{2} - \frac{e_{-1}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} - \frac{e_{-1}}{2} \\ 0 & 0 & 0 & e_{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & e_{-1} \end{pmatrix}$$

In fine, the matrix $PQ^{-1}He_C$ is a fundamental matrix of solutions to the 2-Mahler system with matrix $A(z)$.

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