HAHN SERIES AND MAHLER EQUATIONS: HEIGHT GAP THEOREM

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Abstract. This paper is a companion to our previous work [FR25] and focuses on the asymptotic growth of the logarithmic Weil height of the coefficients of Hahn series solutions to Mahler equations. Extending recent results of Adamczewski, Bell, and Smertnig on power series solutions of Mahler equations, we show that five distinct types of growth behavior can occur. We further prove that three of these growth behaviors correspond to specific automaticity or regularity properties, reminiscent of the notions of φ-biautomaticity and quasi-φ-biautomaticity introduced by Kedlaya in his description of the algebraic closure of the field of Laurent series over an algebraically closed field of positive characteristic. As a consequence, we obtain an alternative formulation of the purity theorem *à la* André and Chudnovsky–Chudnovsky for Mahler equations established in [FR25]. In particular, we show that the minimal Mahler equation associated with a regular Hahn series admits a basis of solutions whose elements also satisfy the corresponding regularity property.

Contents

1. Introduction

We let $p \geqslant 2$ be an integer. The present work takes its source in the recent paper [ABS23] in which Adamczewski, Bell and Smertnig conduct an in-depth study of the asymptotic growth of the coefficients of p-Mahler power series, as measured by their logarithmic Weil height. By p-Mahler power series, we mean a power series $f(z) \in \overline{\mathbb{Q}}[[z]]$ with coefficients in the

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field of algebraic numbers $\overline{\mathbb{Q}}$ satisfying a linear p-Mahler equation, that is a linear functional equation of the form

$$a_0(z)f(z) + a_1(z)f(z^p) + \cdots + a_m(z)f(z^{p^m}) = 0 \tag{1}$$

where $a_0(z), \ldots, a_m(z)$ belong to the field

$$\mathbb{K}_\infty = \overline{\mathbb{Q}}(z^{\frac{1}{\star}}) = \bigcup_{k \in \mathbb{Z}_{\geqslant 1}} \overline{\mathbb{Q}}(z^{\frac{1}{k}})$$

of ramified rational functions and are such that $a_0(z)a_m(z) \neq 0$. More generally, in the rest of this paper, by p-Mahler Laurent series (resp. Puiseux series, Hahn series, *etc.*), we will mean a Laurent series (resp. Puiseux series, Hahn series, *etc.*) satisfying a functional equation of the form (1). The study of Mahler equations has developed into a rich and rapidly evolving area of research. The main result of [ABS23] is the following striking height gap theorem.

Theorem 1.1 ([ABS23, Theorem 1.1])**.** *The sequence of coefficients* $(f_k)_{k \geqslant 0}$ *of any p-Mahler power series* $f(z) = \sum_{k \in \mathbb{Z}_{\geqslant 0}} f_k z^k \in \overline{\mathbb{Q}}[[z]]$ *satisfies one of the following mutually exclusive properties:*

$(\mathcal{O}\Omega_1)$ $\mathscr{h}(f_k) \in \mathcal{O} \cap \Omega(k)$;
$(\mathcal{O}\Omega_2)$ $\mathscr{h}(f_k) \in \mathcal{O} \cap \Omega(\log^2 k)$;
$(\mathcal{O}\Omega_3)$ $\mathscr{h}(f_k) \in \mathcal{O} \cap \Omega(\log k)$;
$(\mathcal{O}\Omega_4)$ $\mathscr{h}(f_k) \in \mathcal{O} \cap \Omega(\log \log k)$;
$(\mathcal{O}\Omega_5)$ $\mathscr{h}(f_k) \in \mathcal{O}(1)$.

We have denoted by $\mathscr{h}(\alpha)$ the logarithmic Weil height of $\alpha \in \overline{\mathbb{Q}}$, *i.e.*, $\mathscr{h}(\alpha) = \log H(\alpha)$ where $H(\alpha)$ is the Weil height of α. We refer to Section 3 for details and references, and simply note here that these quantities are intended to capture the arithmetic "size" of α. For instance, if $\gamma = \frac{a}{b}$ is a rational number written in lowest terms, then $\mathscr{h}(\gamma) = \log \max\{|a|, |b|\}$. Moreover, for any $(a_k)_{k \geqslant 0}, (b_k)_{k \geqslant 0} \in \mathbb{R}^{\mathbb{Z}_{\geqslant 0}}$, the notation $a_k \in \mathcal{O}(b_k)$ means that there exists $C > 0$ such that, for all but finitely many $k \in \mathbb{Z}_{\geqslant 0}$, we have $|a_k| \leqslant C|b_k|$ and the notation $a_k \in \Omega(b_k)$ means that there exists $c > 0$ such that, for infinitely many $k \in \mathbb{Z}_{\geqslant 0}$, we have $|a_k| > c|b_k|$. Last, the notation $a_k \in \mathcal{O} \cap \Omega(b_k)$ means that $a_k \in \mathcal{O}(b_k)$ and $a_k \in \Omega(b_k)$.

In addition to Theorem 1.1, Adamczewski, Bell and Smertnig have established the remarkable fact that the p-Mahler power series satisfying $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ or $(\mathcal{O}\Omega_5)$ correspond to the generating series of classical classes of sequences. Their result can be stated as follows (we refer to Section 8 for further details and the relevant definitions).

Theorem 1.2 ([ABS23, Theorem 1.2])**.** *Let* $f(z) \in \overline{\mathbb{Q}}[[z]]$ *be a p-Mahler power series. The following hold:*

- $f(z)$ *is p-regular if and only if* $f(z)$ *satisfies* $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ *or* $(\mathcal{O}\Omega_5)$;
- $f(z)$ *is p-automatic if and only if* $f(z)$ *satisfies* $(\mathcal{O}\Omega_5)$.

Now, let us turn to the content of the present article.

1.1. p-Mahler equations and Hahn series.

Theorem 1.1 can be easily extended to Puiseux series; see Proposition 5.2. However, much more complicated series arise in the context of p-Mahler equations. This can already be observed on the following simple order 1 (inhomogeneous) p-Mahler equation:

$$f(z^p) - f(z) = z^{-1}. \tag{2}$$

Indeed, it is easily seen that (2) has no Puiseux series solution but, if we permit series with arbitrary denominators in their exponents, we find the following solution:

$$\sum_{k \geqslant 1} z^{-\frac{1}{p^k}}. \tag{3}$$

The latter series belongs to the field

$$\mathscr{H} = \overline{\mathbb{Q}}((z^{\mathbb{Q}}))$$

of Hahn series with coefficients in $\overline{\mathbb{Q}}$ and value group $\mathbb{Q}$; see Section 2 for details and references. It turns out that Hahn series are central to the theory of Mahler equations as shown by the following result: the difference field $(\mathscr{H}, \phi_p)$, where ϕ_p is the field automorphism of $\mathscr{H}$ sending $f(z)$ on $f(z^p)$, has a difference ring extension $(\mathscr{R}, \phi_p)$ with field of constants $\mathscr{R}^{\phi_p} = \{f \in \mathscr{R} \mid \phi_p(f) = f\}$ equal to $\overline{\mathbb{Q}}$ such that

- for any $c \in \overline{\mathbb{Q}}^{\times}$, there exists $e_c \in \mathscr{R}$ which is not a zero divisor satisfying $\phi_p(e_c) = c e_c$;
- there exists $\ell \in \mathscr{R}$ satisfying $\phi_p(\ell) = \ell + 1$;
- any p-Mahler equation of the form (1) has m $\overline{\mathbb{Q}}$-linearly independent solutions $y_1, \ldots, y_m \in \mathscr{R}$ of the form

$$y_i = \sum_{(c,j) \in \Xi} f_{i,c,j} e_c \ell^j \tag{4}$$

where the sum has finite support included in $\Xi = \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}$ and the $f_{i,c,j} \in \mathscr{H}$ are p-Mahler Hahn series.

See [Roq21, Roq24].

This result motivates the following definition.

Definition 1.3. *We will call generalized p-Mahler Hahn series*[1] *any element y of $\mathscr{R}$ of the form*

$$y = \sum_{(c,j) \in \Xi} f_{c,j} e_c \ell^j \tag{5}$$

where the sum is finite and the $f_{c,j} \in \mathscr{H}$ are p-Mahler Hahn series.

1.2. Height gap theorem for p-Mahler Hahn series.

The preceding discussion naturally raises the question of how the coefficients of the p-Mahler Hahn series behave. In light of Section 1.1, and in particular the decomposition (4), answering this question would provide a thorough understanding of the behavior of the coefficients of solutions to p-Mahler equations. Our main result with this respect is the following extension of Theorem 1.1 to Hahn series.

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[1] In [FR25], we introduced the notion of generalized p-Mahler series. As explained in Section 6 these series coincide with the generalized p-Mahler Hahn series introduced here. However, since we focus on the coefficients of the Hahn series involved in (4), we prefer to refer to them as generalized p-Mahler *Hahn* series in this paper.

Theorem 1.4. *Any p-Mahler Hahn series* $f(z) = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$ *satisfies one of the following mutually exclusive properties:*

$(\mathcal{O}\Omega_1)$ $h(f_\gamma) \in \mathcal{O} \cap \Omega(H(\gamma))$;
$(\mathcal{O}\Omega_2)$ $h(f_\gamma) \in \mathcal{O} \cap \Omega(\log^2 H(\gamma))$;
$(\mathcal{O}\Omega_3)$ $h(f_\gamma) \in \mathcal{O} \cap \Omega(\log H(\gamma))$;
$(\mathcal{O}\Omega_4)$ $h(f_\gamma) \in \mathcal{O} \cap \Omega(\log\log H(\gamma))$;
$(\mathcal{O}\Omega_5)$ $h(f_\gamma) \in \mathcal{O}(1)$.

In the previous result, we have used the following natural extensions of the notations $\mathcal{O}$ and Ω recalled above. For any $(a_\gamma)_{\gamma \in \mathbb{Q}}, (b_\gamma)_{\gamma \in \mathbb{Q}} \in \mathbb{R}^{\mathbb{Q}}$, the notation $a_\gamma \in \mathcal{O}(b_\gamma)$ means that there exists $C > 0$ such that, for all but finitely many $\gamma \in \mathbb{Q}$, we have $|a_\gamma| \leqslant C|b_\gamma|$ and the notation $a_\gamma \in \Omega(b_\gamma)$ means that there exists $c > 0$ such that, for infinitely many $\gamma \in \mathbb{Q}$, we have $|a_\gamma| > c|b_\gamma|$. Last, the notation $a_\gamma \in \mathcal{O} \cap \Omega(b_\gamma)$ means that $a_\gamma \in \mathcal{O}(b_\gamma)$ and $a_\gamma \in \Omega(b_\gamma)$.

Remark 1.5. *1. When* $f(z) \in \overline{\mathbb{Q}}[[z]]$, *each condition* $(\mathcal{O}\Omega_r)$ *of Theorem 1.4 is equivalent to the corresponding condition* $(\mathcal{O}\Omega_r)$ *of Theorem 1.1. Indeed, for any* $\gamma = k \in \mathbb{Z}_{\geqslant 1}$, *we have* $H(\gamma) = k$.

2. However, the following example shows that, when dealing with Hahn series, the quantity $H(\gamma)$ *cannot be replaced by the usual absolute value* $|\gamma|$ *in the growth conditions* $(\mathcal{O}\Omega_r)$ *of Theorem 1.4. Consider the slight variant of (3) given by*

$$g(z) = \sum_{\gamma \in \mathbb{Q}} g_\gamma z^\gamma = \sum_{k \geqslant 1} 2^k z^{1-\frac{1}{p^k}} \in \mathscr{H},$$

which is solution of

$$y(z^p) = 2z^{p-1} y(z) + 2z^{p-1}.$$

The Weil height of the coefficients of $g(z)$ *is given by*

$$h(g_{1-\frac{1}{p^k}}) = h(2^k) = k \log(2).$$

Their growth is unrelated to the growth of the usual Archimedean absolute value of $1 - \frac{1}{p^k}$, *since the latter is approximately equal to* 1. *Rather, it is related to the growth of the denominator of* $1 - \frac{1}{p^k}$, *to which the Weil height is sensitive.*

1.3. Quasi-p-regular and quasi-p-automatic Hahn series.

In light of Theorem 1.2, it is natural to ask whether the p-Mahler Hahn series satisfying $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ or $(\mathcal{O}\Omega_5)$ have a particular origin. It turns out that this is indeed the case. The key concepts, introduced Section 8, are those of quasi-p-automatic and quasi-p-regular Hahn series, which extend to Hahn series the notions of p-automatic and p-regular power series, and are reminiscent of the quasi-φ-biautomatic Hahn series introduced by Kedlaya over fields of positive characteristic in [Ked17]. Their origin lies in theoretical computer science. Heuristically, quasi-p-regular Hahn series are those that, after a gauge transformation, can be produced by a weighted finite automaton reading the base-p expansion of the elements of $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. Among them, quasi-$p$-automatic Hahn series are precisely those for which one may take the automaton to be deterministic. We prove the following.

Theorem 1.6. *Let $f(z)$ be a p-Mahler Hahn series. The following hold:*

- (Reg) *$f(z)$ is quasi-p-regular if and only if $f(z)$ satisfies $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ or $(\mathcal{O}\Omega_5)$;*
- (Aut) *$f(z)$ is quasi-p-automatic if and only if $f(z)$ satisfies $(\mathcal{O}\Omega_5)$.*

Note that a Hahn series satisfies $(\mathcal{O}\Omega_5)$ if and only if its coefficients belong to a finite set. Thus (Aut) can be restated as follows: *a p-Mahler Hahn series is quasi-p-automatic if and only if its coefficients belong to a finite set.*

Furthermore, it is actually possible to distinguish between the three growth behaviors $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ and $(\mathcal{O}\Omega_5)$ in Case (Reg) of Theorem 1.6 by considering the minimal linear representation associated with the sequence of coefficients of $f(z)$; see Section 8.7 for further details.

Remark 1.7. *Roughly speaking, Theorem 1.6 asserts that p-Mahler Hahn series exhibiting "special" growth properties must have a "special" origin. This phenomenon is in keeping with the spirit of the Bombieri–Dwork conjecture, which predicts that the minimal differential equation satisfied by a G-function is of geometric origin.*

1.4. **Purity theorem.** Theorem 1.4 reveals five $\mathcal{O}$-growth conditions for p-Mahler Hahn series: we say that $f = \sum_\gamma f_\gamma z^\gamma \in \mathscr{H}$ satisfies

- $(\mathcal{O}_1)$ if $\mathcal{h}(f_\gamma) \in \mathcal{O}(H(\gamma))$;
- $(\mathcal{O}_2)$ if $\mathcal{h}(f_\gamma) \in \mathcal{O}(\log^2 H(\gamma))$;
- $(\mathcal{O}_3)$ if $\mathcal{h}(f_\gamma) \in \mathcal{O}(\log H(\gamma))$;
- $(\mathcal{O}_4)$ if $\mathcal{h}(f_\gamma) \in \mathcal{O}(\log\log H(\gamma))$;
- $(\mathcal{O}_5)$ if $\mathcal{h}(f_\gamma) \in \mathcal{O}(1)$.

We extend these growth conditions to the generalized p-Mahler Hahn series as follows.

Definition 1.8. *Consider a generalized p-Mahler Hahn series*

$$y = \sum_{(c,j)\in\Xi} f_{c,j} e_c \ell^j \in \mathscr{R} \tag{6}$$

where the sum is finite and the $f_{c,j} \in \mathscr{H}$ are p-Mahler Hahn series. We say that y satisfies $(\mathscr{H} - \mathcal{O}_r)$ for some $r \in \{1, \ldots, 5\}$ if the $f_{c,j} \in \mathscr{H}$ involved in (6) satisfy $(\mathcal{O}_r)$.

It follows from Theorem 1.4 that any generalized p-Mahler Hahn series satisfies $(\mathscr{H} - \mathcal{O}_1)$. Therefore, the five growth conditions $(\mathscr{H} - \mathcal{O}_1)$ to $(\mathscr{H} - \mathcal{O}_5)$ induce the following filtration on the set of generalized p-Mahler series:

$$\begin{aligned}
&\{\text{generalized } p\text{-Mahler series}\}\\
&\quad = \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{H} - \mathcal{O}_1)\}\\
&\qquad \supsetneq \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{H} - \mathcal{O}_2)\}\\
&\qquad\quad \supsetneq \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{H} - \mathcal{O}_3)\}\\
&\qquad\qquad \supsetneq \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{H} - \mathcal{O}_4)\}\\
&\qquad\qquad\quad \supsetneq \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{H} - \mathcal{O}_5)\}.
\end{aligned}$$

We are now ready to state our purity theorem guaranteeing that the membership of a generalized p-Mahler series to one of the three largest pieces of this filtration propagates to all other generalized Mahler Hahn series solutions of its minimal Mahler equation.

Theorem 1.9 (Purity Theorem)**.** *Assume that a generalized p-Mahler Hahn series y satisfies ($\mathscr{H} - \mathcal{O}_r$) for some $r \in \{1, 2, 3\}$. Then, the minimal p-Mahler equation of y over $\mathbb{K}_\infty$ has a full basis of generalized p-Mahler Hahn series solutions satisfying ($\mathscr{H} - \mathcal{O}_r$).*

Combining Theorem 1.9 with Theorem 1.6, we immediately obtain the following result.

Corollary 1.10. *Let $f \in \mathscr{H}$ be a quasi-p-regular Hahn series. Then, its minimal p-Mahler equation over $\mathbb{K}_\infty$ has a full basis of solutions $y_1, \dots, y_m$ of the form*

$$y_i = \sum_{(c,j)\in\Xi} f_{i,c,j}\, e_c \ell^j$$

where the sum is finite and the $f_{i,c,j}$ are quasi-p-regular Hahn series.

1.5. **Proof strategy and organization of the paper.** Section 2 starts with a brief review of Hahn series and continues with the description of an explicit basis $(\xi_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\boldsymbol{\Lambda}_{\mathrm{st}}}$ of the module of p-Mahler Hahn series over the ring of p-Mahler Puiseux series introduced in [FR25]. This basis will serve as a crucial tool in the proofs of our main results. Indeed, we will deduce Theorem 1.4 as a special case of a more precise statement, namely Theorem 5.1, and the proof of the latter may be summarized as follows.

- We first establish the result for p-Mahler Puiseux series as a direct consequence of Theorem 1.1.
- We then prove it for the elements of the basis $(\xi_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\boldsymbol{\Lambda}_{\mathrm{st}}}$. In this step, we no longer appeal to Theorem 1.1, but instead rely on a novel height gap theorem for multi-recurrence sequences established in Section 4, which is of independent interest.
- Next, we prove Theorem 5.1 when $f(z)$ is the product of a p-Mahler Puiseux series and one of the Hahn series $\xi_{\boldsymbol{\omega}}$.
- Finally, we prove it in full generality by carefully analyzing the intersections of the supports of such products.

As preliminaries to this program, Section 3 recalls some facts about the logarithmic Weil height of algebraic numbers. In Section 4, we establish the height gap theorem for multi-recurrence sequences mentioned above. Finally, Section 5 is devoted to the proof of Theorem 5.1 (from which Theorem 1.4 follows immediately), according to the strategy outlined above.

In Section 6, using Theorem 5.1, we prove Theorem 1.9 by showing that it is equivalent to the purity theorem from [FR25].

Finally, Sections 7 and 8 are devoted to notions of automaticity and regularity for sequences and Hahn series, and contain the proof of Theorem 1.6. As in the proof of Theorem 1.4, we proceed in stages for proving Theorem 1.6: first for Puiseux series, then for the elements of the basis $(\xi_{\boldsymbol{\omega}})_{\boldsymbol{\omega}\in\boldsymbol{\Lambda}_{\mathrm{st}}}$, and finally for products of both. The general case then follows easily.

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2. p-Mahler Hahn series

2.1. **Generalities on Hahn series.** We recall that an element of the field of Hahn series $\mathscr{H}$ is an $f = (f_\gamma)_{\gamma \in \mathbb{Q}} \in \overline{\mathbb{Q}}^{\mathbb{Q}}$ whose support

$$\operatorname{supp}(f) = \{\gamma \in \mathbb{Q} \mid f_\gamma \neq 0\}$$

is well-ordered (*i.e.*, any nonempty subset of $\operatorname{supp}(f)$ has a least element) with respect to the restriction to $\operatorname{supp}(f)$ of the usual order on $\mathbb{Q}$. Such an element of $\mathscr{H}$ is usually (and will be) denoted by

$$f = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma.$$

The sum and product of two elements $f = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma$ and $g = \sum_{\gamma \in \mathbb{Q}} g_\gamma z^\gamma$ of $\mathscr{H}$ are given by

$$f + g = \sum_{\gamma \in \mathbb{Q}} (f_\gamma + g_\gamma) z^\gamma \quad \text{and} \quad fg = \sum_{\gamma \in \mathbb{Q}} \left(\sum_{\gamma' + \gamma'' = \gamma} f_{\gamma'} g_{\gamma''} \right) z^\gamma.$$

The fact that the supports of f and g are well-ordered implies that there are only finitely many $(\gamma', \gamma'') \in \mathbb{Q} \times \mathbb{Q}$ such that $\gamma' + \gamma'' = \gamma$ and $f_{\gamma'} g_{\gamma''} \neq 0$. Thus, the sums $\sum_{\gamma' + \gamma'' = \gamma} f_{\gamma'} g_{\gamma''}$ are meaningful.

The field $\mathscr{H}$ of Hahn series contains the field

$$\mathscr{P} = \overline{\mathbb{Q}}((z^{\frac{1}{\star}})) = \bigcup_{k \in \mathbb{Z}_{\geqslant 1}} \overline{\mathbb{Q}}((z^{\frac{1}{k}}))$$

of Puiseux series as a subfield but it is much bigger. A typical example of Hahn series which is not a Puiseux series is given by (3).

We shall now introduce a family of Hahn series that will be central to this paper.

2.2. **The Hahn series $\xi_{\boldsymbol{\omega}}$.** Consider the following sets

$$\boldsymbol{\Lambda} = \bigcup_{t \in \mathbb{Z}_{\geqslant 0}} \boldsymbol{\Lambda}_t \quad \text{where} \quad \boldsymbol{\Lambda}_t = \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^\times)^t \times \mathbb{Q}_{>0}^t.$$

Notation 2.1. Throughout this paper, unless stated otherwise, we let $\boldsymbol{\omega}$ be an element of $\boldsymbol{\Lambda}$. We will set $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a})$ with $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{Z}_{\geqslant 0}^t$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t) \in (\overline{\mathbb{Q}}^\times)^t$ and $\boldsymbol{a} = (a_1, \ldots, a_t) \in \mathbb{Q}_{>0}^t$ where t is the unique element of $\mathbb{Z}_{\geqslant 0}$ such that $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_t$. When several elements of $\boldsymbol{\Lambda}$ are considered simultaneously, we will use variants such as $\boldsymbol{\omega}' = (\boldsymbol{\alpha}', \boldsymbol{\lambda}', \boldsymbol{a}')$ with $\boldsymbol{\alpha}' = (\alpha'_1, \ldots, \alpha'_{t'}) \in \mathbb{Z}_{\geqslant 0}^{t'}$, $\boldsymbol{\lambda}' = (\lambda'_1, \ldots, \lambda'_{t'}) \in (\overline{\mathbb{Q}}^\times)^{t'}$ and $\boldsymbol{a}' = (a'_1, \ldots, a'_{t'}) \in \mathbb{Q}_{>0}^{t'}$.

If $t \in \mathbb{Z}_{\geqslant 1}$, then, for any $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_t$, we consider the Hahn series defined by

$$\xi_{\boldsymbol{\omega}}(z) = \sum_{k_1, \ldots, k_t \geqslant 1} k_1^{\alpha_1} \cdots k_t^{\alpha_t} \lambda_1^{k_1} \lambda_2^{k_1 + k_2} \cdots \lambda_t^{k_1 + \cdots + k_t} z^{-\frac{a_1}{p^{k_1}} - \frac{a_2}{p^{k_1 + k_2}} - \cdots - \frac{a_t}{p^{k_1 + k_2 + \cdots + k_t}}} \in \mathscr{H}.$$

When $t = 0$, the sets $\mathbb{Z}^t_{\geqslant 0}$, $(\overline{\mathbb{Q}}^\times)^t$ and $\mathbb{Q}^t_{>0}$ have just one element, namely the empty vector $()$, so $\mathbf{\Lambda}_0 = \{((), (), ())\}$ and we set

$$\xi_{((),(),())}(z) = 1.$$

These are well-defined p-Mahler Hahn series; see [FR25, Section 2].

2.3. Standard decomposition of p-Mahler series.

The interest of the Hahn series $\xi_{\boldsymbol{\omega}}$ in our context lies in the fact, shown in [FR25], that any p-Mahler Hahn series is a linear combination with coefficients in the ring of p-Mahler Puiseux series of the $\xi_{\boldsymbol{\omega}}$ with $\boldsymbol{\omega} \in \mathbf{\Lambda}$. However, such a decomposition is not unique in general. To solve this problem, we introduced in *loc. cit.* the set

$$\mathbf{\Lambda}_{\mathrm{st}} = \bigcup_{t \in \mathbb{Z}_{\geqslant 0}} \mathbb{Z}^t_{\geqslant 0} \times (\overline{\mathbb{Q}}^\times)^t \times \mathbb{N}^t_{(p)} \subset \mathbf{\Lambda}$$

where $\mathbb{N}_{(p)}$ denotes the set of positive rational numbers whose denominator is coprime with p and whose numerator is not divisible by p. We have the following result.

Proposition 2.2 ([FR25, Proposition 25]). *Any p-Mahler Hahn series $f(z) \in \mathscr{H}$ can be uniquely written as a finite sum of the form*

$$f = \sum_{\boldsymbol{\omega} \in \mathbf{\Lambda}_{\mathrm{st}}} f_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}} \tag{7}$$

where $(f_{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \mathbf{\Lambda}_{\mathrm{st}}}$ is a family of p-Mahler Puiseux series.

In other words, the family of p-Mahler Hahn series $(\xi_{\boldsymbol{\omega}})_{\boldsymbol{\omega} \in \mathbf{\Lambda}_{\mathrm{st}}}$ form a basis of the module of p-Mahler Hahn series over the ring of p-Mahler Puiseux series.

3. LOGARITHMIC WEIL HEIGHT, CONDITIONS $(\mathscr{O}\Omega_r)$ AND $(\mathscr{O}_r)$

3.1. Logarithmic Weil height of an algebraic number.

We normalize the non-trivial absolute values on number fields as in [Wal00]. Precisely, for the archimedean place of $\mathbb{Q}$, we use the usual absolute value and, for any prime number p, we normalize the p-adic absolute value by $|p|_p = 1/p$. For a number field K and a place w of K extending a place v of $\mathbb{Q}$, let

$$|\alpha|_{\mathsf{w}} = |\,\mathrm{N}_{K_{\mathsf{w}}/\mathbb{Q}_{\mathsf{v}}}(\alpha)|_{\mathsf{v}}^{1/[K_{\mathsf{w}}:\mathbb{Q}_{\mathsf{v}}]},$$

where K_{w} and $\mathbb{Q}_{\mathsf{v}}$ denote the completions of K and $\mathbb{Q}$ with respect to the places w and v respectively. The set of places M_K on K satisfies the product formula, *i.e.*, for any $\alpha \in K^\times$, we have

$$\prod_{\mathsf{w} \in M_K} |\alpha|_{\mathsf{w}}^{[K_{\mathsf{w}}:\mathbb{Q}_{\mathsf{v}}]} = 1.$$

The absolute Weil height of $\alpha \in \overline{\mathbb{Q}}$ is defined by

$$H(\alpha) = \prod_{\mathsf{w} \in M_K} \max\{1, |\alpha|_{\mathsf{w}}^{[K_{\mathsf{w}}:\mathbb{Q}_{\mathsf{v}}]}\}$$

where K is any number field containing α. The value $H(\alpha)$ does not depend on the choice of the number field K. For instance, for $\alpha = a/b \in \mathbb{Q}^\times$ with a, $b \in \mathbb{Z}$, $b \neq 0$, and $\gcd(a, b) = 1$, we have

$$H(\alpha) = H(a/b) = \max\{|a|, |b|\}. \tag{8}$$

The logarithmic absolute Weil height of $\alpha \in \overline{\mathbb{Q}}$ is defined by

$$\hbar(\alpha) = \log H(\alpha).$$

The only properties we shall use in the rest of the paper are the following:

$$\begin{array}{rcl}\hbar(\alpha^n) & = & |n|\hbar(\alpha) \text{ for any } n \in \mathbb{Z}, \text{ when } \alpha \neq 0, \\ \hbar(\alpha\beta) & \leqslant & \hbar(\alpha) + \hbar(\beta), \\ \hbar(\alpha+\beta) & \leqslant & \log(2) + \hbar(\alpha) + \hbar(\beta), \\ \hbar(\alpha) & = & 0 \quad \text{if and only if } \alpha \text{ is 0 or a root of unity.}\end{array} \tag{9}$$

For further details on these and other properties of the Weil height we refer the reader to [Wal00].

3.2. The conditions $(\mathcal{O}\Omega_r)$ and $(\mathcal{O}_r)$.

Consider the maps on $\mathbb{Q}$ defined as follows:

- $\hbar_1 : \gamma \mapsto H(\gamma)$,
- $\hbar_2 : \gamma \mapsto \log^2 H(\gamma)$,
- $\hbar_3 : \gamma \mapsto \log H(\gamma)$,
- $\hbar_4 : \gamma \mapsto \log\log H(\gamma)$ for $\gamma \notin \{-1, 0, 1\}$, and $\hbar_4(\gamma) = 0$ otherwise,
- $\hbar_5 : \gamma \mapsto 1$.

With these notations, for any $r \in \{1, \ldots, 5\}$, the condition $(\mathcal{O}\Omega_r)$ introduced in Theorem 1.4 of Section 1 becomes:

$$(\mathcal{O}\Omega_r) \quad \hbar(f_\gamma) \in \mathcal{O} \cap \Omega(\hbar_r(\gamma))$$

and the condition $(\mathcal{O}_r)$ introduced in Section 1.4 becomes:

$$(\mathcal{O}_r) \quad \hbar(f_\gamma) \in \mathcal{O}(\hbar_r(\gamma)).$$

The following lemmas will be used several times in the rest of the paper.

Lemma 3.1. *Consider $r \in \{1, \ldots, 5\}$, $\nu \in \mathbb{Q}$ and $d \in \mathbb{Q}^\times$. There exist $c_1, c_2 > 0$ such that, for all but finitely many $\gamma \in \mathbb{Q}$,*

$$c_1\hbar_r(\gamma) \leqslant \hbar_r(d\gamma + \nu) \leqslant c_2\hbar_r(\gamma).$$

Proof. Let

$$c_0 := \log(2) + \hbar(d) + \hbar(\nu) = \log(2) + \hbar(d^{-1}) + \hbar(-\nu).$$

Using (9), we get, for all $\gamma \in \mathbb{Q}$,

$$\begin{aligned}\hbar(d\gamma+\nu) &\leqslant \log(2) + \hbar(d\gamma) + \hbar(\nu) \\ &\leqslant \log(2) + \hbar(d) + \hbar(\gamma) + \hbar(\nu) = \hbar(\gamma) + c_0\,.\end{aligned}$$

Similarly, we have, for all $\gamma \in \mathbb{Q}$,

$$\hbar(d\gamma+\nu) \geqslant \hbar(\gamma) - \hbar(d^{-1}) - \log(2) - \hbar(-\nu) = \hbar(\gamma) - c_0$$

So, we have, for all $\gamma \in \mathbb{Q}$,

$$\hbar(\gamma) - c_0 \leqslant \hbar(d\gamma+\nu) \leqslant \hbar(\gamma) + c_0. \tag{10}$$

But, for all but finitely many $\gamma \in \mathbb{Q}$ we have $\hbar_r(\gamma) = f_r(\hbar(\gamma))$ where $f_1(x) = e^x$, $f_2(x) = x^2$, $f_3(x) = x$, $f_4(x) = \log x$ and $f_5(x) = 1$. Applying f_r to (10), we get, for all but finitely many $\gamma \in \mathbb{Q}$,

$$f_r(\hbar(\gamma) - c_0) \leqslant \hbar_r(d\gamma+\nu) \leqslant f_r(\hbar(\gamma) + c_0).$$

As, for any $M > 0$, the set of $\gamma \in \mathbb{Q}$ such that $\mathscr{h}(\gamma) \leqslant M$ is finite, in order to conclude, it is sufficient to prove that there exist $c_1, c_2 > 0$ such that, for all $x > 0$ large enough, $f_r(x - c_0) \geqslant c_1 f_r(x)$ and $f_r(x + c_0) \leqslant c_2 f_r(x)$. The latter property is clearly true. □

Lemma 3.2. *Consider $r \in \{1, \ldots, 5\}$, $\nu \in \mathbb{Q}$, $d \in \mathbb{Q}_{>0}$ and $f \in \mathscr{H}$. The following properties are equivalent:*

(a) *$f \in \mathscr{H}$ satisfies $(\mathcal{O}\Omega_r)$;*
(b) *$z^\nu f(z^d)$ satisfies $(\mathcal{O}\Omega_r)$.*

Proof. Let us first prove that (b) implies (a). So, we assume that $g(z) = z^\nu f(z^d)$ satisfies $(\mathcal{O}\Omega_r)$. Setting $f(z) = \sum_\gamma f_\gamma z^\gamma$ and $g(z) = \sum_\gamma g_\gamma z^\gamma$, we have, for any $\gamma \in \mathbb{Q}$, $f_\gamma = g_{d\gamma+\nu}$. On the one hand, since g satisfies $(\mathcal{O}\Omega_r)$, we have

$$\mathscr{h}(f_\gamma) = \mathscr{h}(g_{d\gamma+\nu}) \in \mathcal{O}(\mathscr{h}_r(d\gamma + \nu))$$

and it follows from Lemma 3.1 that

$$\mathscr{h}(f_\gamma) \in \mathcal{O}(\mathscr{h}_r(\gamma)).$$

On the other hand, since g satisfies $(\mathcal{O}\Omega_r)$, there exists $c_0 > 0$ such that, for infinitely many $\gamma \in \mathbb{Q}$,

$$\mathscr{h}(f_\gamma) = \mathscr{h}(g_{d\gamma+\nu}) \geqslant c_0 \mathscr{h}_r(d\gamma + \nu).$$

Furthermore, Lemma 3.1 ensures that there exists $c_1 > 0$ such that, for all but finitely many $\gamma \in \mathbb{Q}$, we have $\mathscr{h}_r(d\gamma + \nu) > c_1 \mathscr{h}_r(\gamma)$. Thus, for infinitely many $\gamma \in \mathbb{Q}$,

$$\mathscr{h}(f_\gamma) = \mathscr{h}(g_{d\gamma+\nu}) \geqslant c_0 \mathscr{h}_r(d\gamma + \nu) \geqslant c_0 c_1 \mathscr{h}_r(\gamma)\,.$$

It follows that f satisfies $(\mathcal{O}\Omega_r)$. This concludes the proof that (b) implies (a).

Let us now prove that (a) implies (b). So, we assume that f satisfies $(\mathcal{O}\Omega_r)$. Setting $g(z) = z^\nu f(z^d)$, we have that $z^{-\frac{\nu}{d}} g(z^{\frac{1}{d}}) = f(z)$ satisfies $(\mathcal{O}\Omega_r)$. By the first part of the proof applied with f instead of g, $-\frac{\nu}{d}$ instead of ν and $\frac{1}{d}$ instead of d, we get that g satisfies $(\mathcal{O}\Omega_r)$ as well. □

Similarly, we have:

Lemma 3.3. *Consider $r \in \{1, \ldots, 5\}$, $\nu \in \mathbb{Q}$, $d \in \mathbb{Q}_{>0}$ and $f \in \mathscr{H}$. The following properties are equivalent:*

(1) $f \in \mathscr{H}$ satisfies $(\mathcal{O}_r)$;
(2) $z^\nu f(z^d)$ satisfies $(\mathcal{O}_r)$.

4. A HEIGHT GAP THEOREM FOR MULTI-RECURRENCE SEQUENCES

This section is devoted to the proof of a height gap theorem for multi-recurrence sequences over $\overline{\mathbb{Q}}$. We will use this result to derive Property (ii) of Theorem 5.1, which classifies the Hahn series $\boldsymbol{\xi}_{\boldsymbol{\omega}}$ according to the conditions $(\mathcal{O}\Omega_r)$ in terms of certain properties of the tuple $\boldsymbol{\omega}$. A multi-recurrence sequence is a map of the following form

$$\begin{array}{rcl} F : (\mathbb{Z}_{\geqslant 0})^t & \to & \overline{\mathbb{Q}} \\ (k_1, \ldots, k_t) & \mapsto & \sum_{i=1}^s P_i(k_1, \ldots, k_t)\theta_{i,1}^{k_1} \cdots \theta_{i,t}^{k_t} \end{array} \tag{11}$$

where

- $s \in \mathbb{Z}_{\geqslant 0}$ and $t \in \mathbb{Z}_{\geqslant 1}$;
- $P_1, \ldots, P_s \in \overline{\mathbb{Q}}[X_1, \ldots, X_t] \backslash \{0\}$;
- the t-uples $\boldsymbol{\theta}_1 := (\theta_{1,1}, \ldots, \theta_{1,t}), \ldots, \boldsymbol{\theta}_s := (\theta_{s,1}, \ldots, \theta_{s,t}) \in (\overline{\mathbb{Q}}^\times)^t$ are pairwise distinct.

These maps have been the subject of many papers; see [FH22] and the references therein. Our main result with respect to these sequences is the following height gap theorem.

Theorem 4.1 (Height gap theorem for multi-recurrence sequences)**.** *Any multi-recurrence sequence F of the form (11) satisfies one of the following properties:*

(Lin) $\mathcal{h}(F(k_1, \ldots, k_t)) \in \mathcal{O} \cap \Omega(k_1 + \cdots + k_t)$ *if one of the $\theta_{i,j}$ is not a root of unity;*

(Log) $\mathcal{h}(F(k_1, \ldots, k_t)) \in \mathcal{O} \cap \Omega(\log(k_1 + \cdots + k_t))$ *if each $\theta_{i,j}$ is a root of unity and some P_i is non-constant;*

(Bnd) $\mathcal{h}(F(k_1, \ldots, k_t)) \in \mathcal{O}(1)$ *if each $\theta_{i,j}$ is a root of unity and each P_i is constant.*

In the previous result, given two maps $u, v : (\mathbb{Z}_{\geqslant 0})^t \to \mathbb{R} \cup \{-\infty\}$, the notation $u(k_1, \ldots, k_t) \in \mathcal{O}(v(k_1, \ldots, k_t))$ means that there exists $c > 0$ such that, for all but finitely many $(k_1, \ldots, k_t) \in (\mathbb{Z}_{\geqslant 0})^t$, $|u(k_1, \ldots, k_t)| \leqslant c|v(k_1, \ldots, k_t)|$. The notation $u(k_1, \ldots, k_t) \in \Omega(v(k_1, \ldots, k_t))$ means that there exists $c > 0$ such that, for any $C \geqslant 0$, there exists $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$ such that $|u(k_1, \ldots, k_t)| \geqslant c|v(k_1, \ldots, k_t)|$; it is equivalent to require that there exists $c > 0$ such that, for any $C \geqslant 0$, there exist infinitely many t-uples $(k_1, \ldots, k_t) \in (\mathbb{Z}_{\geqslant C})^t$ such that $|u(k_1, \ldots, k_t)| \geqslant c|v(k_1, \ldots, k_t)|$. Last, the notation $u(k_1, \ldots, k_t) \in \mathcal{O} \cap \Omega(v(k_1, \ldots, k_t))$ means that $u(k_1, \ldots, k_t) \in \mathcal{O}(v(k_1, \ldots, k_t))$ and $u(k_1, \ldots, k_t) \in \Omega(v(k_1, \ldots, k_t))$.

Before proving Theorem 4.1, we establish a non-vanishing result for multi-recurrence sequences.

4.1. Non-vanishing of multi-recurrence sequences.

The decomposition (11) of a multi-recurrence sequence is unique up to reordering the terms, as the following result implies.

Lemma 4.2. *A multi-recurrence sequence (11) is null if and only if $s = 0$.*

Proof. Lemma 2.2 of [Sch03] states that, for any positive integer t, the maps of the form

$$\begin{array}{ccc} \mathbb{Z}^t & \to & \overline{\mathbb{Q}} \\ (k_1, \ldots, k_t) & \mapsto & k_1^{a_1} \cdots k_t^{a_t} \theta_1^{k_1} \cdots \theta_t^{k_t}, \end{array}$$

with $a_1, \ldots, a_t \in \mathbb{Z}_{\geqslant 0}$ and $\theta_1, \ldots, \theta_t \in \overline{\mathbb{Q}}^\times$, are linearly independent over $\overline{\mathbb{Q}}$. The same argument applies verbatim to their restrictions to $(\mathbb{Z}_{\geqslant 0})^t$, yielding the present lemma. □

We deduce the following.

Lemma 4.3. *Let F be a multi-recurrence sequence of the form (11) with $s \geqslant 1$. Then, for any $C \geqslant 0$, there exist $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$ such that $F(k_1, \ldots, k_t) \neq 0$.*

Since the values of a multi-recurrence sequence generate a finite field extension of $\mathbb{Q}$, one may reformulate this as follows: for any non-zero multi-recurrence sequence F, one has $h(F(k_1,\ldots,k_t)) \in \Omega(1)$.

Proof. We prove the contrapositive statement. So, we suppose that there exists $C \geqslant 0$ such that, for all $k_1,\ldots,k_t \in \mathbb{Z}_{\geqslant C}$, $F(k_1,\ldots,k_t) = 0$. Consider the map $G : (\mathbb{Z}_{\geqslant 0})^t \to \overline{\mathbb{Q}}$ defined by

$$\begin{aligned} G(k_1,\ldots,k_t) &= F(k_1+C,\ldots,k_t+C) \\ &= \sum_{i=1}^{s} P_i(k_1+C,\ldots,k_t+C)(\theta_{i,1}\cdots\theta_{i,t})^C \theta_{i,1}^{k_1}\cdots\theta_{i,t}^{k_t}. \end{aligned}$$

This multi-recurrence sequence is equal to 0 by hypothesis. By Lemma 4.2, for all $i \in \{1,\ldots,s\}$, the polynomial

$$P_i(X_1+C,\ldots,X_t+C)(\theta_{i,1}\cdots\theta_{i,t})^C \in \overline{\mathbb{Q}}[X_1,\ldots,X_t]$$

is null. Thus, the polynomials $P_1,\ldots,P_s$ are null, whence a contradiction. ☐

4.2. **Proof of Theorem 4.1.** Theorem 4.1 combines two types of estimates: lower bounds (the Ω-estimates) and upper bounds (the $\mathcal{O}$-estimates). These estimates are established in the following two sections and are then combined in Section 4.2.3 to prove Theorem 4.1.

4.2.1. *Lower bounds.*

Lemma 4.4. *Let F be a multi-recurrence sequence of the form (11) with $s \geqslant 1$. The following properties hold:*

(a) $h(F(k_1,\ldots,k_t)) \in \Omega(k_1+\cdots+k_t)$ *if one of the $\theta_{i,j}$ is not a root of unity;*

(b) $h(F(k_1,\ldots,k_t)) \in \Omega(\log(k_1+\cdots+k_t))$ *if one of the P_i is non-constant.*

Proof. We only prove (a), the proof of (b) being similar.

Let us first prove (a) in the case $t = 1$. Let $\Theta \subset \overline{\mathbb{Q}}^\times$ be a finite set of algebraic numbers, not all roots of unity, and let

$$c_\Theta := \min\{h(\theta) \mid \theta \in \Theta \text{ not a root of unity}\} > 0.$$

Let $F : \mathbb{Z}_{\geqslant 0} \to \overline{\mathbb{Q}}$ be defined by

$$F(k) = P_1(k)\theta_1^k + \cdots + P_s(k)\theta_s^k$$

for some $P_1,\ldots,P_s \in \overline{\mathbb{Q}}[X]\backslash\{0\}$ and some pairwise distinct $\theta_1,\ldots,\theta_s \in \Theta$ that are not all roots of unity. We prove by induction on $\Delta(F) := \sum_{i=1}^{s}(\deg P_i + 1)$ that, for infinitely many $k \in \mathbb{Z}_{\geqslant 0}$,

$$h(F(k)) \geqslant \frac{c_\Theta}{3^{\Delta(F)}} k. \tag{12}$$

Base case. If $\Delta(F) = 1$, then $s = 1$, $P_1 \in \overline{\mathbb{Q}}^\times$, θ_1 is not a root of unity and $F(k) = P_1\theta_1^k$. It follows from (9) that

$$h(F(k)) = h(P_1\theta_1^k) \geqslant h(\theta_1^k) - h(P_1) = kh(\theta_1) - h(P_1) \geqslant kc_\Theta - h(P_1).$$

Hence, (12) holds for all $k \in \mathbb{Z}_{\geqslant 0}$ large enough.

Inductive step. Suppose $\Delta(F) \geqslant 2$. Up to renumbering, we can assume that θ_s is not a root of unity. We consider the recurrence sequence $G : \mathbb{Z}_{\geqslant 0} \to \overline{\mathbb{Q}}$ defined by

$$G(k) = F(k+1) - \theta_1 F(k) = \sum_{j=1}^{s} Q_j(k)\theta_j^k$$

where $Q_j(X) = \theta_j P_j(X+1) - \theta_1 P_j(X) \in \overline{\mathbb{Q}}[X]$. Note that, for any $j \in \{2, \dots, s\}$, we have $\deg Q_j = \deg P_j$. Furthermore, $Q_1 = 0$ if $\deg P_1 = 0$ and $\deg Q_1 = \deg P_1 - 1$ otherwise. Therefore, we have $\Delta(G) = \Delta(F) - 1$. By induction, there exists an infinite set $\mathcal{K}$ such that, for all $k \in \mathcal{K}$, we have

$$\mathcal{h}(G(k)) \geqslant \frac{c_\Theta}{3^{\Delta(F)-1}} k.$$

Moreover, using (9), we see that

$$\mathcal{h}(F(k+1)) + \mathcal{h}(\theta_1) + \mathcal{h}(F(k)) + \log(2) \geqslant \mathcal{h}(G(k)).$$

Combining the previous two inequalities, we get that, for any $k \in \mathcal{K}$ large enough, one of the following holds:

$$\text{either } \mathcal{h}(F(k)) \geqslant \frac{c_\Theta}{3^{\Delta(F)}} k \quad \text{ or } \quad \mathcal{h}(F(k+1)) \geqslant \frac{c_\Theta}{3^{\Delta(F)}} k.$$

Thus (12) holds for infinitely many k. This concludes the inductive step and the proof of (a) in the case $t = 1$.

Let us now prove (a) for an arbitrary $t \geqslant 2$. Let $\Theta = \{\theta_{1,t}, \dots, \theta_{s,t}\}$. Up to renumbering, we can and will suppose that $\theta_{s,t}$ is not a root of unity. Let $C > 0$. Lemma 4.3 ensures that there exist $k_1, \dots, k_t \in \mathbb{Z}_{\geqslant C}$ such that

$$\sum_{\substack{i \in \{1,\dots,s\} \\ \text{s.t. } \theta_{i,t} = \theta_{s,t}}} P_i(k_1, \dots, k_t)\theta_{i,1}^{k_1} \cdots \theta_{i,t}^{k_t} = \left(\sum_{\substack{i \in \{1,\dots,s\} \\ \text{s.t. } \theta_{i,t} = \theta_{s,t}}} P_i(k_1, \dots, k_t)\theta_{i,1}^{k_1} \cdots \theta_{i,t-1}^{k_{t-1}} \right) \theta_{s,t}^{k_t} \neq 0. \tag{13}$$

Fix such $k_1, \dots, k_{t-1} \in \mathbb{Z}_{\geqslant C}$ and consider the recurrence sequence $G : \mathbb{Z}_{\geqslant 0} \to \overline{\mathbb{Q}}$ defined by

$$G(k) = F(k_1, \dots, k_{t-1}, k).$$

Gathering the terms of the sum (11) defining F according to the value of $\theta_{i,t}$, we see that

$$G(k) = \sum_{j=1}^{u} Q_j(k)\eta_j^k$$

for some pairwise distinct $\eta_1 = \theta_{s,t}, \eta_2, \dots, \eta_u \in \Theta$ and some $Q_1, \dots, Q_u \in \overline{\mathbb{Q}}[X] \backslash \{0\}$. The fact that we may take $\eta_1 = \theta_{s,t}$ with $Q_1 \neq 0$ follows from (13). Let $\Delta(G)$ and c_Θ be defined as in the case $t = 1$ treated at the beginning of the proof. One may find an integer Δ, not depending on $k_1, \dots, k_{t-1}$ nor on C, such that $\Delta(G) \leqslant \Delta$. It follows from (12) that, for infinitely many $k \in \mathbb{Z}_{\geqslant C}$,

$$\mathcal{h}(F(k_1, \dots, k_{t-1}, k)) = \mathcal{h}(G(k)) \geqslant \frac{c_\Theta}{3^\Delta} k. \tag{14}$$

Consider an integer $k \geqslant k_1 + \ldots + k_{t-1}$ satisfying (14). Then, we have

$$\mathit{h}(F(k_1, \ldots, k_{t-1}, k)) \geqslant \frac{c_\Theta}{2 \times 3^\Delta}(k_1 + \cdots + k_{t-1} + k).$$

This proves (a). ☐

4.2.2. *Upper bounds.*

Lemma 4.5. *Let F be a multi-recurrence sequence of the form (11) with $s \geqslant 1$. The following properties hold:*

(i) $\mathit{h}(F(k_1, \ldots, k_t)) = \mathcal{O}(k_1 + \cdots + k_t)$ *in any case;*
(ii) $\mathit{h}(F(k_1, \ldots, k_t)) = \mathcal{O}(\log(k_1 + \cdots + k_t))$ *if all $\theta_{i,j}$ are roots of unity;*
(iii) $\mathit{h}(F(k_1, \ldots, k_t)) = \mathcal{O}(1)$ *if all $\theta_{i,j}$ are roots of unity and all the polynomials P_i are constants.*

Proof. Given a polynomial $P \in \overline{\mathbb{Q}}[X_1, \ldots, X_t]$, it follows from (9) that

$$\mathit{h}(P(k_1, \ldots, k_t)) = \mathcal{O}(\log(\max\{k_1, \ldots, k_t\})) \,. \tag{15}$$

Using (9) and (11), we also have

$$\mathit{h}(F(k_1, \ldots, k_t)) \leqslant (s-1)\log(2) + \sum_{i=1}^{s} \mathit{h}(P_i(k_1, \ldots, k_t)) + \sum_{i=1}^{s}\sum_{j=1}^{t} k_j \mathit{h}(\theta_{i,j}). \tag{16}$$

These two inequalities combined imply (i).

Now, suppose that all $\theta_{i,j}$ are roots of unity. Then $\mathit{h}(\theta_{i,j}) = 0$ for all i, j and it follows from (15) and (16) that

$$\mathit{h}(F(k_1, \ldots, k_t)) = \mathcal{O}(\log(\max\{k_1, \ldots, k_t\})) = \mathcal{O}(\log(k_1 + \cdots + k_t)),$$

which implies (ii).

Last, suppose that all $\theta_{i,j}$ are roots of unity and that the polynomials $P_1, \ldots, P_s$ are constants. Then, $F(k_1, \ldots, k_t)$ only takes a finite number of distinct values. Thus (iii) holds. ☐

4.2.3. *Proof of Theorem 4.1.* Suppose that one of the $\theta_{i,j}$ is not a root of unity. Then, from (a) of Lemma 4.4 and (i) of Lemma 4.5 we have $\mathit{h}(F(k_1, \ldots, k_t)) \in \mathcal{O} \cap \Omega(k_1 + \cdots + k_t)$.

Suppose that all $\theta_{i,j}$ are roots of unity but that one of the P_i is non-constant. Then, from (b) of Lemma 4.4 and (ii) of Lemma 4.5 we have $\mathit{h}(F(k_1, \ldots, k_t)) \in \mathcal{O} \cap \Omega(\log(k_1 + \cdots + k_t))$.

Finally, suppose that all $\theta_{i,j}$ are roots of unity and that all P_i are constant. Then, from (iii) of Lemma 4.5 we have $\mathit{h}(F(k_1, \ldots, k_t)) \in \mathcal{O}(1)$.

Since any multi-recurrence sequence falls into the scope of one of these three mutually exclusive cases, Theorem 4.1 holds.

5. A REFINED VERSION OF THEOREM 1.4

Theorem 1.4 asserts that any p-Mahler Hahn series $f(z) \in \mathscr{H}$ satisfies one of the mutually exclusive conditions ($\mathcal{O}\Omega_1$)–($\mathcal{O}\Omega_5$). This follows directly from the following more precise result, whose proof is the main objective of this section.

Theorem 5.1. *Consider a p-Mahler Hahn series $f(z) \in \mathscr{H}$ and its standard decomposition*

$$f = \sum_{\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}} f_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}} \tag{17}$$

where $(f_{\boldsymbol{\omega}})_{\boldsymbol{\Lambda}_{\mathrm{st}}}$ is family of p-Mahler Puiseux series with finite support (see Section 2.3). Then, we have

- (i) *every $f_{\boldsymbol{\omega}}$ satisfies one of the mutually exclusive conditions $(\mathcal{O}\Omega_1)$ to $(\mathcal{O}\Omega_5)$;*
- (ii) *every $\xi_{\boldsymbol{\omega}}$ satisfies one of the mutually exclusive conditions $(\mathcal{O}\Omega_3)$ to $(\mathcal{O}\Omega_5)$;*
- (iii) *f satisfies $(\mathcal{O}\Omega_r)$ with*

$$r = \min\left\{ s \in \{1, \ldots, 5\} \;\middle|\; \begin{array}{l} f_{\boldsymbol{\omega}} \textit{ or } \xi_{\boldsymbol{\omega}} \textit{ satisfies } (\mathcal{O}\Omega_s) \\ \textit{for some } \boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}} \textit{ such that } f_{\boldsymbol{\omega}} \neq 0 \end{array} \right\}.$$

Properties (i), (ii) and (iii) are respectively proved in Section 5.1, Section 5.2 and Section 5.3.

5.1. Proof of Property (i) of Theorem 5.1.

Property (i) of Theorem 5.1 is a direct consequence of the following result, which itself follows easily from Theorem 1.1.

Proposition 5.2. *Any p-Mahler Puiseux series $f(z) \in \mathscr{P}$ satisfies one of the mutually exclusive properties $(\mathcal{O}\Omega_1)$ to $(\mathcal{O}\Omega_5)$.*

Proof. Since $f(z)$ is a p-Mahler Puiseux series, there exist $\nu \in \mathbb{Z}$ and $d \in \mathbb{Z}_{>0}$ such that $z^\nu f(z^d)$ is a p-Mahler power series. Theorem 1.1 ensures that $z^\nu f(z^d)$ satisfies one of the mutually exclusive properties $(\mathcal{O}\Omega_1)$ to $(\mathcal{O}\Omega_5)$. Lemma 3.2 guarantees that $f(z)$ satisfies one of the mutually exclusive properties $(\mathcal{O}\Omega_1)$ to $(\mathcal{O}\Omega_5)$ as well. ☐

5.2. Proof of Property (ii) of Theorem 5.1.

Recall that we let

$$\boldsymbol{\Lambda}_{\mathrm{st}} = \bigcup_{t \in \mathbb{Z}_{\geqslant 0}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^\times)^t \times \mathbb{N}_{(p)}^t$$

where

$$\mathbb{N}_{(p)} = \{a/b \mid a, b \in \mathbb{Z}_{>0},\ p \nmid a,\ \gcd(b,p) = 1\},$$

and that for $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}$, we write $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a})$ with

$$\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{Z}_{\geqslant 0}^t,\ \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t) \in (\overline{\mathbb{Q}}^\times)^t,\ \boldsymbol{a} = (a_1, \ldots, a_t) \in \mathbb{N}_{(p)}^t.$$

In this section, we prove the following result, which yields Property (ii) of Theorem 5.1 when $C = 1$. Allowing an arbitrary value of C will be useful in subsequent applications.

Proposition 5.3. *Let $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}$ with $\boldsymbol{\omega} \neq ((), (), ())$ and let $C \in \mathbb{Z}_{\geqslant 1}$. The Hahn series*

$$\xi_{\boldsymbol{\omega}, C}(z) = \sum_{k_1, \ldots, k_t \geqslant C} k_1^{\alpha_1} \cdots k_t^{\alpha_t} \lambda_1^{k_1} \cdots \lambda_t^{k_1 + \cdots + k_t} z^{-\frac{a_1}{p^{k_1}} - \cdots - \frac{a_t}{p^{k_1 + \cdots + k_t}}}$$

satisfies:

- $(\mathcal{O}\Omega_3)$ *if and only if one of the coordinates of $\boldsymbol{\lambda}$ is not a root of unity;*

- $(\mathcal{O}\Omega_4)$ *if and only if the coordinates of* $\boldsymbol{\lambda}$ *are roots of unity and* $\boldsymbol{\alpha}$ *is not the zero vector;*
- $(\mathcal{O}\Omega_5)$ *if and only if the coordinates of* $\boldsymbol{\lambda}$ *are roots of unity and* $\boldsymbol{\alpha}$ *is the zero vector.*

Proof. We may write

$$\xi_{\boldsymbol{\omega},C}(z) = \sum_{k_1,\ldots,k_t \in \mathbb{Z}_{\geqslant C}} F(k_1,\ldots,k_t) z^{\gamma_{k_1,\ldots,k_t}}$$

with

$$F(k_1,\ldots,k_t) = k_1^{\alpha_1}\cdots k_t^{\alpha_t}\lambda_1^{k_1}\cdots\lambda_t^{k_1+\cdots+k_t}$$

and

$$\gamma_{k_1,\ldots,k_t} = -\frac{a_1}{p^{k_1}} - \frac{a_2}{p^{k_1+k_2}} - \cdots - \frac{a_t}{p^{k_1+k_2+\cdots+k_t}}. \tag{18}$$

Note that F is a multi-recurrence sequence since

$$F(k_1,\ldots,k_t) = P(k_1,\ldots,k_t)\theta_1^{k_1}\cdots\theta_t^{k_t} \tag{19}$$

where

$$P(k_1,\ldots,k_t) = k_1^{\alpha_1}\cdots k_t^{\alpha_t}$$

and, for each $j \in \{1,\ldots,t\}$,

$$\theta_j = \lambda_j \cdots \lambda_t.$$

Lemma 5.4 below ensures that the rational numbers $\gamma_{k_1,\ldots,k_t}$ are pairwise distinct when $k_1,\ldots,k_t$ vary over $\mathbb{Z}_{\geqslant 1}$. So, we have

$$\xi_{\boldsymbol{\omega},C}(z) = \sum_{\gamma\in\mathbb{Q}} \zeta_\gamma z^\gamma$$

where

$$\zeta_\gamma = \begin{cases} F(k_1,\ldots,k_t) & \text{if } \gamma = \gamma_{k_1,\ldots,k_t} \text{ for some } k_1,\ldots,k_t \in \mathbb{Z}_{\geqslant C};\\ 0 & \text{else.}\end{cases} \tag{20}$$

We split the rest of the proof into three cases.

Case 1: one of the λ_j *is not a root of unity.* Then, one of the θ_j is not a root of unity and it follows from case (Lin) of Theorem 4.1 that

$$h(F(k_1,\ldots,k_t)) = \mathcal{O}(k_1+\cdots+k_t)$$

and that there exists $c > 0$ such that

$$h(F(k_1,\ldots,k_t)) \geqslant c(k_1+\cdots+k_t)$$

for infinitely many $(k_1,\ldots,k_t) \in (\mathbb{Z}_{\geqslant C})^t$. Combining this with (20) above and (22) from Lemma 5.5 below, we obtain that $\xi_{\boldsymbol{\omega},C}$ satisfies $(\mathcal{O}\Omega_3)$.

Case 2: all the λ_i *are roots of unity and at least one of the* α_i *is nonzero.* In this case, all the θ_j are roots of unity and the polynomial P is non-constant. It follows from case (Log) of Theorem 4.1 that

$$h(F(k_1,\ldots,k_t)) = \mathcal{O}(\log(k_1+\cdots+k_t))$$

and that there exists $c > 0$ such that

$$h(F(k_1,\ldots,k_t)) \geqslant c(\log(k_1+\cdots+k_t))$$

for infinitely many $(k_1,\ldots,k_t) \in (\mathbb{Z}_{\geqslant C})^t$. Combining this with (20) above and (22) from Lemma 5.5 below, we obtain that $\xi_{\boldsymbol{\omega},C}$ satisfies $(\mathcal{O}\Omega_4)$.

Case 3: all the λ_i are roots of unity and all the α_i are equal to zero. In this case, the coefficients ζ_γ take only a finite number of values and, hence, we get that $\xi_{\boldsymbol{\omega},C}$ satisfies $(\mathcal{O}\Omega_5)$. ☐

It remains to prove the following two lemmas.

Lemma 5.4. *For any $t \in \mathbb{Z}_{\geqslant 1}$ and $\boldsymbol{a} \in \mathbb{N}_{(p)}^t$, the rational numbers $\gamma_{k_1,\ldots,k_t}$ defined by (18) are pairwise distinct when $k_1,\ldots,k_t$ vary over $\mathbb{Z}_{\geqslant 1}$.*

Proof. Assume to the contrary that

$$\gamma_{k_1,\ldots,k_t} = \gamma_{\ell_1,\ldots,\ell_t} \tag{21}$$

for distinct t-uples $(k_1,\ldots,k_t) \in \mathbb{Z}_{\geqslant 1}^t$ and $(\ell_1,\ldots,\ell_t) \in \mathbb{Z}_{\geqslant 1}^t$. The t-uples $(k_1, k_1+k_2,\ldots,k_1+\cdots+k_t)$ and $(\ell_1,\ell_1+\ell_2,\ldots,\ell_1+\cdots+\ell_t)$ are distinct. Let i be the greatest element of $\{1,\ldots,t\}$ such that $k_1+\cdots+k_i \neq \ell_1+\cdots+\ell_i$. We can assume that $k_1+\cdots+k_i < \ell_1+\cdots+\ell_i$. Then, (21) implies that

$$\frac{a_1}{p^{k_1}} + \frac{a_2}{p^{k_1+k_2}} + \cdots + \frac{a_i}{p^{k_1+k_2+\cdots+k_i}} = \frac{a_1}{p^{\ell_1}} + \frac{a_2}{p^{\ell_1+\ell_2}} + \cdots + \frac{a_i}{p^{\ell_1+\ell_2+\cdots+\ell_i}}.$$

Multiplying the latter equality by $p^{\ell_1+\ell_2+\cdots+\ell_i}$, we obtain that the numerator of a_i is a multiple of p. This in contradiction with the fact that a_i belongs to $\mathbb{N}_{(p)}$. ☐

Lemma 5.5. *For any $t \in \mathbb{Z}_{\geqslant 1}$ and $\boldsymbol{a} \in \mathbb{N}_{(p)}^t$, there exist $c_0, c_1 > 0$ such that, for all $k_1,\ldots,k_t \in \mathbb{Z}_{\geqslant 1}$, the height of the rational number $\gamma_{k_1,\ldots,k_t}$ defined by (18) satisfies*

$$c_0 p^{k_1+\ldots+k_t} \leqslant H(\gamma_{k_1,\ldots,k_t}) \leqslant c_1 p^{k_1+\cdots+k_t}. \tag{22}$$

Proof. Using that $a_1,\ldots,a_t \in \mathbb{N}_{(p)}$, it is easily seen that there exist $u_1,\ldots,u_t \in \mathbb{Z}_{>0}$, not divisible by p, and $v \in \mathbb{Z}_{>0}$ coprime to p such that, for all $k_1,\ldots,k_t \in \mathbb{Z}_{\geqslant 1}$,

$$\gamma_{k_1,\ldots,k_t} = -\frac{u_1 p^{k_2+k_3+\cdots+k_t} + u_2 p^{k_3+\cdots+k_t} + \cdots + u_t}{v p^{k_1+k_2+\cdots+k_t}}. \tag{23}$$

The upper bound for $H(\gamma_{k_1,\ldots,k_t})$ in (22) then follows by taking

$$c_1 = \max\{v, u_1+u_2+\cdots+u_t\}.$$

To prove the lower bound, it clearly suffices to show that the following family of gcds is bounded:

$$\Big(\gcd\Big(\alpha_{k_2,\ldots,k_t},\, p^{k_1+k_2+\cdots+k_t}\Big)\Big)_{k_1,\ldots,k_t\geqslant 1},$$

where

$$\alpha_{k_2,\ldots,k_t} = u_1 p^{k_2+k_3+\cdots+k_t} + u_2 p^{k_3+\cdots+k_t} + \cdots + u_t.$$

To this end, it is clearly enough to prove that, for every prime factor q of p, the following family of q-adic valuations is bounded:

$$\big(v_q(\alpha_{k_2,\ldots,k_t})\big)_{k_2,\ldots,k_t\geqslant 1}. \tag{24}$$

This is true when $t = 1$ because $\alpha_{k_2,\ldots,k_t} = u_t$ is a nonzero constant. Let us now assume that $t \geqslant 2$ and suppose, for contradiction, that the family (24) is unbounded. Then there exists a sequence $(\boldsymbol{k}_n)_{n\geqslant 0} = (k_{n,2},\dots,k_{n,t})$ in $(\mathbb{Z}_{\geqslant 1})^{t-1}$ such that

$$v_q(\alpha_{\boldsymbol{k}_n}) \xrightarrow[n\to+\infty]{} +\infty.$$

By passing to a subsequence, we may assume that, for each $i \in \{2,\dots,t\}$, the sequence $(k_{n,i})_{n\geqslant 0}$ is either constant or increasing. Let i_0 be the largest index in $\{2,\dots,t\}$ for which $(k_{n,i_0})_{n\geqslant 0}$ is increasing. Such an i_0 exists, since otherwise the sequence $(\boldsymbol{k}_n)_{n\geqslant 0}$ would be constant. Consider the equality

$$\alpha_{\boldsymbol{k}_n} - \left(u_1 p^{k_{n,2}+\cdots+k_{n,t}} + \cdots + u_{i_0-1}p^{k_{n,i_0}+\cdots+k_{n,t}}\right) \\ = u_{i_0}p^{k_{n,i_0+1}+\cdots+k_{n,t}} + \cdots + u_t. \tag{25}$$

The q-adic valuation of the left-hand side of (25) tends to $+\infty$ as $n \to +\infty$, whereas the right-hand side is a nonzero constant, yielding a contradiction. □

5.3. **Proof of Property (iii) of Theorem 5.1.** Property (iii) of Theorem 5.1 will be proved at the end of this section, after establishing several preliminary results.

We begin with the following observation: in general, the coefficient of $z^{\gamma+\gamma'}$ in the product of two Hahn series g and h is not equal to the product of the coefficients of z^γ in g by the coefficient of $z^{\gamma'}$ in h. Nevertheless, in the particular case where $h = \xi_{\boldsymbol{\omega}}$ with $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}$ and $\boldsymbol{\omega} \neq ((),(),())$, and $g \in \mathscr{P}$, we show in Lemma 5.7 below that a (weak) form of this property does hold.

Definition 5.6. *We denote by* $\trianglelefteq$ *the partial order on* $\mathscr{H}$ *defined, for any* $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$ *and* $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma \in \mathscr{H}$, *by*

$$g \trianglelefteq f \quad \Leftrightarrow \quad \forall \gamma \in \operatorname{supp} g, \quad f_\gamma = g_\gamma.$$

Lemma 5.7. *Let* $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}$ *with* $\boldsymbol{\omega} \neq ((),(),())$. *Set* $\xi_{\boldsymbol{\omega}} =: \sum_{\gamma\in\mathbb{Q}} \zeta_\gamma z^\gamma$. *For any* $g \in \mathscr{P}$, *for infinitely many* $\gamma \in \operatorname{supp}(\xi_{\boldsymbol{\omega}})$, *we have* $\zeta_\gamma z^\gamma g \trianglelefteq g\xi_{\boldsymbol{\omega}}$.

To prove this Lemma, we will use the following result.

Lemma 5.8 ([FR25, Lemma 27]). *For all* $s,t \in \mathbb{Z}_{\geqslant 0}$ *such that* $s \geqslant t \geqslant 0$, *for all* $\boldsymbol{a} = (a_1,\dots,a_s) \in \mathbb{N}^s_{(p)}$, *for all* $\boldsymbol{b} = (b_1,\dots,b_t) \in \mathbb{N}^t_{(p)}$, *for all* $d \in \mathbb{Z}_{\geqslant 1}$, *there exists* $C_{\boldsymbol{a},\boldsymbol{b},d} > 0$ *such that, for all* $\gamma \in \frac{1}{d}\mathbb{Z}$, *for all* $k_1,\dots,k_s,\ell_1,\dots,\ell_t \in \mathbb{Z}_{\geqslant 1}$ *such that* $k_1,\dots,k_s \geqslant C_{\boldsymbol{a},\boldsymbol{b},d}$, *the following properties are equivalent:*

(1) we have

$$\frac{a_1}{p^{k_1}} + \frac{a_2}{p^{k_1+k_2}} + \cdots + \frac{a_s}{p^{k_1+k_2+\cdots+k_s}} = \gamma + \frac{b_1}{p^{\ell_1}} + \frac{b_2}{p^{\ell_1+\ell_2}} + \cdots + \frac{b_t}{p^{\ell_1+\ell_2+\cdots+\ell_t}};$$

(2) we have $\gamma = 0$, $s = t$ *and, for all* $i \in \{1,\dots,t\}$, $a_i = b_i$ *and* $k_i = \ell_i$.

Remark 5.9. *Lemma 5.4 can be seen as a refinement of Lemma 5.8 in the special case* $\boldsymbol{a} = \boldsymbol{b}$ *and* $\gamma = 0$.

Proof of Lemma 5.7. We set $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma \in \mathscr{P}$, and $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma = g\xi_{\boldsymbol{\omega}}$. Thus, for all $\gamma \in \mathbb{Q}$, we have

$$f_\gamma = \sum_{\substack{(\gamma_1,\gamma_2)\in\operatorname{supp}(g)\times\operatorname{supp}(\xi_{\boldsymbol{\omega}})\\ \text{such that } \gamma_1+\gamma_2=\gamma}} g_{\gamma_1}\zeta_{\gamma_2}.$$

Let d be such that $\operatorname{supp}(g) \subset \frac{1}{d}\mathbb{Z}$. Let $C_{\boldsymbol{a},\boldsymbol{a},d} > 0$ be the constant given by Lemma 5.8. We recall that $\operatorname{supp}(\xi_{\boldsymbol{\omega}}) = \{\gamma_{k_1,\ldots,k_t} \mid k_1,\ldots,k_t \in \mathbb{Z}_{\geqslant 1}\}$ where $\gamma_{k_1,\ldots,k_t}$ is given by (18). Our choice of constant $C_{\boldsymbol{a},\boldsymbol{a},d}$ ensures that, for all $\gamma_1 \in \operatorname{supp}(g)$ and all $\gamma_2 = \gamma_{k_1,\ldots,k_t} \in \operatorname{supp}(\xi_{\boldsymbol{\omega}})$ with $k_1,\ldots,k_t \geqslant C_{\boldsymbol{a},\boldsymbol{a},d}$, if

$$\gamma_1 + \gamma_2 = \gamma_1' + \gamma_2'$$

for some $(\gamma_1',\gamma_2') \in \operatorname{supp}(g) \times \operatorname{supp}(\xi_{\boldsymbol{\omega}})$, then $(\gamma_1',\gamma_2') = (\gamma_1,\gamma_2)$. Therefore, for all such $\gamma_2 \in \operatorname{supp}(\xi_{\boldsymbol{\omega}})$, for all $\gamma_1 \in \operatorname{supp}(g)$, we have $f_{\gamma_1+\gamma_2} = g_{\gamma_1}\zeta_{\gamma_2}$ and, hence, $\zeta_{\gamma_2} z^{\gamma_2} g \trianglelefteq f$. Since there are infinitely many such γ_2, this proves the lemma. ☐

We continue with the following observation: two Hahn series g and h being given, there is in general no interesting way to bound from above the heights $H(\gamma_1)$ and $H(\gamma_2)$ in terms of $H(\gamma_1+\gamma_2)$ for all $\gamma_1 \in \operatorname{supp}(g)$ and $\gamma_2 \in \operatorname{supp}(h)$. The following Lemma shows that that such a bound can nevertheless be obtained in the special case where $h = \xi_{\boldsymbol{\omega}}$ with $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}$ and $\boldsymbol{\omega} \neq ((),(),())$, and $g \in \mathscr{P}$.

Lemma 5.10. *Let $g(z) \in \mathscr{P}^{\times}$ and $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}$, $\boldsymbol{\omega} \neq ((),(),())$. There exists $c_1 > 0$ such that, for all but finitely many $\gamma \in \mathbb{Q}$, for all $\gamma_1 \in \operatorname{supp}(g)$ and $\gamma_2 \in \operatorname{supp}(\xi_{\boldsymbol{\omega}})$ such that $\gamma_1+\gamma_2=\gamma$, we have*

$$\max\{H(\gamma_1), H(\gamma_2)\} \leqslant c_1 H(\gamma). \tag{26}$$

Proof. Let $N \in \mathbb{Z}_{\geqslant 1}$ be such that $\operatorname{supp}\xi_{\boldsymbol{\omega}} \subset [-N,0[$ and let $d \in \mathbb{Z}_{\geqslant 1}$ be such that $\operatorname{supp} g \subset \frac{1}{d}\mathbb{Z}$. Then, for any $\gamma \in \mathbb{Q}$, $\gamma_1 \in \operatorname{supp} g$ and $\gamma_2 \in \operatorname{supp}\xi_{\boldsymbol{\omega}}$ such that $\gamma_1+\gamma_2=\gamma$, we have

$$\gamma_1 = \gamma - \gamma_2 \in]\gamma, \gamma+N] \cap \frac{1}{d}\mathbb{Z}.$$

Using (8), this implies that

$$H(\gamma_1) \leqslant d(|\gamma| + N).$$

Therefore, if $H(\gamma) \geqslant N$, then

$$H(\gamma_1) \leqslant d(|\gamma|+N) \leqslant 2dH(\gamma),$$

since we always have $H(\gamma) \geqslant |\gamma|$. Furthermore, as $|\gamma_2| \leqslant N$, we have $H(\gamma_2) \leqslant N\operatorname{den}(\gamma_2)$ where $\operatorname{den}(\gamma_2)$ is the denominator of γ_2. But, as $\gamma_1 \in \frac{1}{d}\mathbb{Z}$, the denominator of $\gamma_2 = \gamma - \gamma_1$ is at most d times the denominator $\operatorname{den}(\gamma)$ of γ. Thus

$$H(\gamma_2) \leqslant N\operatorname{den}(\gamma_2) \leqslant Nd\operatorname{den}(\gamma) \leqslant NdH(\gamma).$$

Thus, (26) holds with $c_1 = \max\{2d, Nd\}$ provided that $H(\gamma) \geqslant N$. Since the latter condition excludes finitely many γ, this proves the lemma. ☐

Last, we mention the following obvious reformulation of the growth properties $(\mathcal{O}\Omega_r)$ for further reference.

Lemma 5.11. *The following properties relative to $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$ and $r \in \{1,\ldots,5\}$ are equivalent:*

1. *f satisfies $(\mathcal{O}\Omega_r)$;*
2. *$\hbar(f_\gamma) = \mathcal{O}(\hbar_r(\gamma))$ and there exists $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma \in \mathscr{H}$ such that $g \trianglelefteq f$ and $\hbar(g_\gamma) = \Omega(\hbar_r(\gamma))$, where $\hbar_r$ is defined as in Section 3.2.*

Proposition 5.12. *Property (iii) of Theorem 5.1 holds when $f = g\xi_{\boldsymbol{\omega}}$ for some p-Mahler Puiseux series $g \in \mathscr{P}^{\times}$ and some $\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}$.*

Proof. When $\boldsymbol{\omega} = ((),(),())$, we have $\xi_{\boldsymbol{\omega}} = 1$ and the result follows from Property (i) of Theorem 5.1.

We now suppose that $\boldsymbol{\omega} \neq ((),(),())$. Property (i) of Theorem 5.1 ensures that g satisfies $(\mathcal{O}\Omega_{q_1})$ for some $q_1 \in \{1,\ldots,5\}$ while Property (ii) of Theorem 5.1 guarantees that $\xi_{\boldsymbol{\omega}}$ satisfies $(\mathcal{O}\Omega_{q_2})$ for some $q_2 \in \{3,4,5\}$. We must prove that f satisfies $(\mathcal{O}\Omega_r)$ with $r = \min\{q_1,q_2\}$.

Since the support of $\xi_{\boldsymbol{\omega}}$ is bounded and since g is a Puiseux series, the number of terms in the product $g\xi_{\boldsymbol{\omega}}$ which contribute to z^{γ} for some $\gamma \in \mathbb{Q}$ is bounded independently of γ, say by $c_2 > 0$. Write $f = \sum_{\gamma} f_{\gamma} z^{\gamma}$, $g = \sum_{\gamma} g_{\gamma} z^{\gamma}$ and $\xi_{\boldsymbol{\omega}} = \sum_{\gamma} \zeta_{\gamma} z^{\gamma}$. From (9), we have

$$\mathcal{h}(f_{\gamma}) \leqslant c_2 \log(2) + 2c_2 \max_{\gamma_1+\gamma_2=\gamma} \{\mathcal{h}(g_{\gamma_1}), \mathcal{h}(\zeta_{\gamma_2})\}\,. \tag{27}$$

It follows that

$$\begin{aligned}\mathcal{h}(f_{\gamma}) &= \mathcal{O}\left(1 + \max_{\gamma_1+\gamma_2=\gamma} \{\mathcal{h}(g_{\gamma_1}), \mathcal{h}(\zeta_{\gamma_2})\}\right)\\ &= \mathcal{O}\left(1 + \max_{\gamma_1+\gamma_2=\gamma} \{\mathcal{h}_{q_1}(\gamma_1), \mathcal{h}_{q_2}(\gamma_2)\}\right)\\ &= \mathcal{O}\left(\max\{\mathcal{h}_{q_1}(\gamma), \mathcal{h}_{q_2}(\gamma)\}\right)\\ &= \mathcal{O}(\mathcal{h}_r(\gamma)),\end{aligned}$$

where the maps $\mathcal{h}_i$ are defined as in Section 3.2. Indeed, the first equality follows from (27), the second one follows from the fact that g and $\xi_{\boldsymbol{\omega}}$ satisfy $(\mathcal{O}\Omega_{q_1})$ and $(\mathcal{O}\Omega_{q_2})$ respectively, the third one follows from Lemma 5.10 and the last equality is obvious. In order to complete the proof, it remains to prove that, if $r < 5$, then $\mathcal{h}(f_{\gamma}) = \Omega(\mathcal{h}_r(\gamma))$. In order to prove this, we distinguish two cases.

Case $r = q_1$. According to Lemma 5.7, there exist $\gamma \in \mathrm{supp}(\xi_{\boldsymbol{\omega}})$ and $\zeta \in \overline{\mathbb{Q}}^{\times}$ such that $\zeta z^{\gamma} g \trianglelefteq f$. Since g satisfies $(\mathcal{O}\Omega_{q_1})$, so does $\zeta z^{\gamma} g$ by Lemma 3.2. Lemma 5.11 guarantees that $\mathcal{h}(f_{\gamma}) = \Omega(\mathcal{h}_{q_1}(\gamma))$.

Case $r = q_2$. Since $\mathrm{supp}\, g \subset \frac{1}{d}\mathbb{Z}$, Lemma 5.8 implies that there exists $C > 0$ such that, for any $k_1,\ldots,k_t \in \mathbb{Z}_{\geqslant C}$, for any $\ell_1,\ldots,\ell_t \in \mathbb{Z}_{\geqslant 0}$ and for any $\gamma, \kappa \in \mathrm{supp}\, g$,

$$\gamma - \frac{a_1}{p^{k_1}} - \cdots - \frac{a_t}{p^{k_1+\cdots+k_t}} \neq \kappa - \frac{a_1}{p^{\ell_1}} - \cdots - \frac{a_t}{p^{\ell_1+\cdots+\ell_t}}$$

unless $\gamma = \kappa$ and, for any $j \in \{1,\ldots,t\}$, $k_j = \ell_j$. We consider the series

$$\xi_{\boldsymbol{\omega},C}(z) = \sum_{k_1,\ldots,k_t \geqslant C} k_1^{\alpha_1} \cdots k_t^{\alpha_t} \lambda_1^{k_1} \cdots \lambda_t^{k_1+\cdots+k_t} z^{-\frac{a_1}{p^{k_1}} - \cdots - \frac{a_t}{p^{k_1+\cdots+k_t}}}$$

already introduced in Proposition 5.3. Our choice of C guarantees that, for any $\gamma \in \mathrm{supp}\, g$, we have $g_{\gamma} z^{\gamma} \xi_{\boldsymbol{\omega},C} \trianglelefteq f$. But, since $\xi_{\boldsymbol{\omega}}$ satisfies $(\mathcal{O}\Omega_{q_2})$, Proposition 5.3 ensures that $\xi_{\boldsymbol{\omega},C}$ satisfies $(\mathcal{O}\Omega_{q_2})$ as well (indeed, for any $s \in \{3,4,5\}$, it implies that $\xi_{\boldsymbol{\omega}} = \xi_{\boldsymbol{\omega},1}$ satisfies $(\mathcal{O}\Omega_s)$ if and only if $\xi_{\boldsymbol{\omega},C}$ satisfies $(\mathcal{O}\Omega_s)$ as these two properties correspond to the same conditions on the parameters $\boldsymbol{\alpha}, \boldsymbol{\lambda}$). Since $g_\gamma \neq 0$, Lemma 3.2 implies that $g_\gamma z^\gamma \xi_{\boldsymbol{\omega},C}$ satisfies $(\mathcal{O}\Omega_{q_2})$ as well. Lemma 5.11 guarantees that $\hbar(f_\gamma) = \Omega(\hbar_{q_2}(\gamma))$. ☐

We are now able to prove Property (iii) of Theorem 5.1 stating that, given a p-Mahler Hahn series $f(z) \in \mathscr{H}$ and its standard decomposition $f = \sum_{\boldsymbol{\omega}\in\boldsymbol{\Lambda}_{\mathrm{st}}} f_{\boldsymbol{\omega}}\xi_{\boldsymbol{\omega}}$, we have that f satisfies $(\mathcal{O}\Omega_r)$ where r is the least integer such that some of the series $f_{\boldsymbol{\omega}}$ or $\xi_{\boldsymbol{\omega}}$ satisfy $(\mathcal{O}\Omega_r)$.

Proof of (iii) of Theorem 5.1. Suppose that $f \neq 0$, otherwise, there is nothing to prove. Write

$$f = \sum_{\boldsymbol{\omega}\in E} f_{\boldsymbol{\omega}}\xi_{\boldsymbol{\omega}}$$

where $E \subset \boldsymbol{\Lambda}_{\mathrm{st}}$ is a non-empty finite set and the $f_{\boldsymbol{\omega}}$ are non-zero p-Mahler Puiseux series. Proposition 5.12 ensures that, for all $\boldsymbol{\omega} \in E$, the product $f_{\boldsymbol{\omega}}\xi_{\boldsymbol{\omega}}$ satisfies $(\mathcal{O}\Omega_{r_{\boldsymbol{\omega}}})$ where

$$r_{\boldsymbol{\omega}} = \min\{s \in \{1,\dots,5\} \mid f_{\boldsymbol{\omega}} \text{ or } \xi_{\boldsymbol{\omega}} \text{ satisfies } (\mathcal{O}\Omega_s)\}.$$

To prove (iii) of Theorem 5.1, we have to prove that $f(z)$ satisfies $(\mathcal{O}\Omega_r)$ with

$$r = \min_{\boldsymbol{\omega}\in E} r_{\boldsymbol{\omega}}.$$

Let us first note that, for all $\boldsymbol{\omega} \in E$, the product $f_{\boldsymbol{\omega}}\xi_{\boldsymbol{\omega}}$ satisfies $(\mathcal{O}\Omega_{r_{\boldsymbol{\omega}}})$, so it satisfies $(\mathcal{O}_{r_{\boldsymbol{\omega}}})$ and, hence, $(\mathcal{O}_r)$. It follows that $f(z)$ satisfies $(\mathcal{O}_r)$ as well. If $r = 5$, there is nothing more to prove. So, from now on, we assume that $r \leqslant 4$ and it remains to prove that

$$\hbar(f_\gamma) = \Omega(\hbar_r(\gamma)), \tag{28}$$

where the maps $\hbar_i$ are defined as in Section 3.2. To prove this, consider $\boldsymbol{\omega}' \in E$ with $r_{\boldsymbol{\omega}'} = r$. Let $q_1 \in \{1,\dots,5\}$ be such that $f_{\boldsymbol{\omega}'}$ satisfies $(\mathcal{O}\Omega_{q_1})$ and let $q_2 \in \{1,\dots,5\}$ be such that $\xi_{\boldsymbol{\omega}'}$ satisfies $(\mathcal{O}\Omega_{q_2})$. As already noticed, Proposition 5.12 ensures that

$$r = r_{\boldsymbol{\omega}'} = \min\{q_1, q_2\}.$$

The rest of the proof is divided according to the following two cases: we will first consider the case $r = q_1$ and then the case $r = q_2$. We will use the following notations. We write $\boldsymbol{\omega}' = (\boldsymbol{\alpha}', \boldsymbol{\lambda}', \boldsymbol{a}')$ with $\boldsymbol{\alpha}' \in \mathbb{Z}_{\geqslant 0}^t$, $\boldsymbol{\lambda}' \in (\overline{\mathbb{Q}}^\times)^t$ and $\boldsymbol{a}' \in \mathbb{N}_{(p)}^t$. We set $\boldsymbol{a}' = (a'_1,\dots,a'_t)$ and we consider the set

$$E' = \{\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in E \mid \boldsymbol{a} = \boldsymbol{a}'\} \subset E,$$

We let d be such that, for all $\boldsymbol{\omega} \in E$, the Puiseux series $f_{\boldsymbol{\omega}}$ belongs to $\overline{\mathbb{Q}}((z^{\frac{1}{d}}))$. Let $C := \max_{\boldsymbol{a}} C(\boldsymbol{a}', \boldsymbol{a}, d)$ where $C(\boldsymbol{a}', \boldsymbol{a}, d)$ is given by Lemma 5.8 and the maximum is taken over the (finite) set of $\boldsymbol{a}$ such that there exists $\boldsymbol{\alpha}, \boldsymbol{\lambda}$ with $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in E$.

Case $r = q_1$. It follows from Lemma 5.8 that, for all $k_1,\dots,k_t \in \mathbb{Z}_{\geqslant C}$, for all $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}') \in E'$, we have

$$f_{\boldsymbol{\omega}}(z) k_1^{\alpha_1}\cdots k_t^{\alpha_t} \lambda_1^{k_1}\lambda_2^{k_1+k_2}\cdots\lambda_t^{k_1+\cdots+k_t} z^{-\frac{a'_1}{p^{k_1}}-\cdots-\frac{a'_t}{p^{k_1+\cdots+k_t}}} \trianglelefteq f_{\boldsymbol{\omega}}(z)\xi_{\boldsymbol{\omega}}(z)\,.$$

Thus, using Lemma 5.8 once again, for all $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$, the Puiseux series

$$\kappa_{k_1,\ldots,k_t}(z) := \sum_{\boldsymbol{\omega}\in E'} f_{\boldsymbol{\omega}}(z) k_1^{\alpha_1} \cdots k_t^{\alpha_t} \lambda_1^{k_1} \lambda_2^{k_1+k_2} \cdots \lambda_t^{k_1+\cdots+k_t}$$

satisfies

$$\kappa_{k_1,\ldots,k_t}(z) z^{-\frac{a'_1}{p^{k_1}}-\cdots-\frac{a'_t}{p^{k_1+\cdots+k_t}}} \trianglelefteq f(z) = \sum_{\boldsymbol{\omega}\in E} f_{\boldsymbol{\omega}}(z)\xi_{\boldsymbol{\omega}}(z).$$

Using Lemma 3.2, we see that, in order to prove (28), it is sufficient to prove that there exist $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$ such that

$$\hslash((\kappa_{k_1,\ldots,k_t})_\gamma) \in \Omega(\hslash_r(\gamma)).$$

In order to prove this, let us first note that

$$\begin{aligned}(29)\quad V := \operatorname{Span}_{\overline{\mathbb{Q}}}\{\kappa_{k_1,\ldots,k_t}(z) \mid k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}\}& \\ = \operatorname{Span}_{\overline{\mathbb{Q}}}\{f_{\boldsymbol{\omega}}(z) \mid \boldsymbol{\omega} \in E'\} =: W.&\end{aligned}$$

Indeed, the inclusion $V \subset W$ is obvious. If the inclusion $V \subset W$ were not an equality, then there would exist a non-zero element φ in the dual W^* of the finite dimensional $\overline{\mathbb{Q}}$-vector space W vanishing on V. Then, $(\varphi(f_{\boldsymbol{\omega}}))_{\boldsymbol{\omega}\in E'}$ would be a family of elements of $\overline{\mathbb{Q}}$, not all zero, such that, for all $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$,

$$\sum_{\boldsymbol{\omega}=(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}')\in E'} \varphi(f_{\boldsymbol{\omega}}) k_1^{\alpha_1} \cdots k_t^{\alpha_t} \lambda_1^{k_1} \lambda_2^{k_1+k_2} \cdots \lambda_t^{k_1+\cdots+k_t} = 0.$$

Since the left-hand side of the latter equality is a multi-recurrence sequence, this would contradict Lemma 4.3. This proves the equality (29).

As all $\kappa_{k_1,\ldots,k_t}(z)$ with $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$ are p-Mahler Puiseux series, property (i) of Theorem 5.1 ensures that they satisfy one of the mutually exclusive properties $(\mathscr{O}\Omega_1)$ to $(\mathscr{O}\Omega_5)$. We claim that there exist $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$ such that $\kappa_{k_1,\ldots,k_t}(z)$ satisfies $(\mathscr{O}\Omega_s)$ for some $s \in \{1, \ldots, r\}$. Otherwise, for all $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$, $\kappa_{k_1,\ldots,k_t}(z)$ would satisfy $(\mathscr{O}_{r+1})$ and, hence, all elements of V would satisfy $(\mathscr{O}_{r+1})$. This would contradict that $f_{\boldsymbol{\omega}'} \in W = V$ satisfies $(\mathscr{O}\Omega_r)$. This proves our claim.

Now, any $\kappa_{k_1,\ldots,k_t}$ as in our previous claim satisfies $\hslash((\kappa_{k_1,\ldots,k_t})_\gamma) \in \Omega(\hslash_s(\gamma))$ for some $s \in \{1, \ldots, r\}$ and, hence, $\hslash((\kappa_{k_1,\ldots,k_t})_\gamma) \in \Omega(\hslash_r(\gamma))$ and this justifies (28).

Case $r = q_2$. Let γ be an element of the support of $f_{\boldsymbol{\omega}'}$. Up to multiplying f by a suitable power of z, we can and will suppose that $\gamma = 0$. It follows from Lemma 5.8 that for all $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$, the coefficient of $z^{-\frac{a'_1}{p^{k_1}}-\cdots-\frac{a'_t}{p^{k_1+\cdots+k_t}}}$ in $f(z)$ is

$$F(k_1, \ldots, k_t) := \sum_{\boldsymbol{\omega}\in E'} (f_{\boldsymbol{\omega}}(z))_0 k_1^{\alpha_1} \cdots k_t^{\alpha_t} \lambda_1^{k_1} \lambda_2^{k_1+k_2} \cdots \lambda_t^{k_1+\cdots+k_t}$$

so that we have

$$g(z) = \sum_{k_1,\ldots,k_t\geqslant C} F(k_1, \ldots, k_t) z^{-\frac{a'_1}{p^{k_1}}-\cdots-\frac{a'_t}{p^{k_1+\cdots+k_t}}} \trianglelefteq f(z).$$

Note that F is a non-zero multi-recurrence sequence. Indeed, in order to write it under the form (11), we consider a partition

$$E' = E_1 \sqcup \cdots \sqcup E_t$$

of E' given by the equivalence classes $E_1, \ldots, E_s$ for the following equivalence relation:

$$(\boldsymbol{\alpha}_0, \boldsymbol{\lambda}_0, \boldsymbol{a}') \sim (\boldsymbol{\alpha}_1, \boldsymbol{\lambda}_1, \boldsymbol{a}') \Leftrightarrow \boldsymbol{\lambda}_0 = \boldsymbol{\lambda}_1 .$$

Then, for any $i \in \{1, \ldots, s\}$, we choose an arbitrary element $\boldsymbol{\omega}$ of E_i and, for any $j \in \{1, \ldots, t\}$, we set $\theta_{i,j} = \lambda_j \cdots \lambda_t$. We also set

$$P_i(k_1, \ldots, k_t) = \sum_{\boldsymbol{\omega} \in E_i} (f_{\boldsymbol{\omega}}(z))_0 k_1^{\alpha_1} \cdots k_t^{\alpha_t} . \tag{30}$$

Then, we have

$$F(k_1, \ldots, k_t) = \sum_{i=s}^{t} P_i(k_1, \ldots, k_t) \theta_{i,1}^{k_1} \cdots \theta_{i,t}^{k_t} .$$

Let i_0 be such that $\boldsymbol{\omega}' \in E_{i_0}$. Then $(f_{\boldsymbol{\omega}'})_0 \neq 0$ by assumption and we deduce that P_{i_0} is nonzero because the $(\alpha_1, \ldots, \alpha_t)$ involved in the right-hand side of (30) are pairwise distinct. We now distinguish two subcases.

Subcase $r = q_2 = 3$. Recall that $\boldsymbol{\omega}' = (\boldsymbol{\alpha}', \boldsymbol{\lambda}', \boldsymbol{a}')$. Since $\xi_{\boldsymbol{\omega}'} = \xi_{\boldsymbol{\omega}',1}$ satisfies $(\mathscr{O}\Omega_3)$, Proposition 5.3 guarantees that $\boldsymbol{\lambda}'$ has a coordinate which is not a root of unity. Then, there exists $j \in \{1, \ldots, t\}$ such that $\theta_{i_0,j}$ is not a root of unity as well and it follows from Lemma 4.4 that there exists $c > 0$ such that for infinitely many $(k_1, \ldots, k_t) \in (\mathbb{Z}_{\geqslant C})^t$,

$$\hbar(F(k_1, \ldots, k_t)) \geqslant c(k_1 + \cdots + k_t) .$$

Thus, using Lemma 5.5, we get $\hbar(g_\gamma) = \Omega(\hbar_3(\gamma))$. Since $g \trianglelefteq f$, it follows from Lemma 5.11 that $\hbar(f_\gamma) = \Omega(\hbar_3(\gamma))$.

Subcase $r = q_2 = 4$. Since $q_2 = 4$, Proposition 5.3 guarantees that $\boldsymbol{\alpha}'$ is nonzero. Thus, the polynomial P_{i_0} is not constant and it follows from Lemma 4.4 that there exists $c > 0$ such that, for infinitely many $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$,

$$\hbar(F(k_1, \ldots, k_t)) \geqslant c \log(k_1 + \cdots + k_t) .$$

Thus, using Lemma 5.5, we get $\hbar(g_\gamma) = \Omega(\hbar_4(\gamma))$. Since $g \trianglelefteq f$, it follows from Lemma 5.11 that $\hbar(f_\gamma) = \Omega(\hbar_4(\gamma))$. ☐

6. Proof of Theorem 1.9

The proof of Theorem 1.9 is given at the end of this section.

In Section 1.4, for each $r \in \{1, \ldots, 5\}$, we introduced the growth condition $(\mathscr{H} - \mathscr{O}_r)$ for a generalized p-Mahler Hahn series, which depends on its decomposition over the ring of p-Mahler Hahn series. In [FR25], we considered another growth condition $(\mathscr{P} - \mathscr{O}_r)$ defined as follows. Combining (4) and (7), we get that any generalized p-Mahler Hahn series $y \in \mathscr{R}$ can be uniquely written as a finite sum of the form

$$y = \sum_{(c,j) \in \Xi} \sum_{\boldsymbol{\omega} \in \boldsymbol{\Lambda}_{\mathrm{st}}} f_{c,j,\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}} e_c \ell^j$$

where the $f_{c,j,\boldsymbol{\omega}} \in \mathscr{P}$ are p-Mahler Puiseux series and $\Xi = \overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}$.

Definition 6.1. *We say that* y *satisfies* $(\mathscr{P}-\mathcal{O}_r)$ *if any* $f_{c,j,\boldsymbol{\omega}}$ *satisfies* $(\mathcal{O}_r)$.

Proposition 6.2. *For any generalized* p*-Mahler Hahn series* $y \in \mathcal{R}$*, for any* $r \in \{1,2,3\}$*, the following properties are equivalent:*

- y *satisfies* $(\mathscr{H}-\mathcal{O}_r)$*;*
- y *satisfies* $(\mathscr{P}-\mathcal{O}_r)$*.*

Proof. It is clearly sufficient to prove the result in the case when $y = f$ is a non-zero p-Mahler Hahn series. So, we consider a p-Mahler Hahn series $f \in \mathscr{H}$ and its standard decomposition

$$f = \sum_{\boldsymbol{\omega} \in E} f_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}}$$

with $f_{\boldsymbol{\omega}}$ some non-zero p-Mahler Puiseux series and $E \subset \boldsymbol{\Lambda}_{\mathrm{st}}$ a non-empty finite set. Consider $r \in \{1,2,3\}$. The result follows from the fact that the following properties are equivalent:

(1) f satisfies $(\mathscr{H}-\mathcal{O}_r)$;
(2) f satisfies $(\mathcal{O}\Omega_s)$ for some $s \in \{r,\ldots,5\}$;
(3) for all $\boldsymbol{\omega} \in E$, $f_{\boldsymbol{\omega}}$ satisfies $(\mathcal{O}\Omega_s)$ for some $s \in \{r,\ldots,5\}$;
(4) for all $\boldsymbol{\omega} \in E$, $f_{\boldsymbol{\omega}}$ satisfies $(\mathcal{O}_r)$;
(5) f satisfies $(\mathscr{P}-\mathcal{O}_r)$.

The equivalence between (1) and (2) follows from Theorem 1.4. The equivalence between (2) and (3) follows from the last assertion of Theorem 5.1 using that the $\xi_{\boldsymbol{\omega}}$ satisfy $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ or $(\mathcal{O}\Omega_5)$ by Property (ii) of Theorem 5.1 and that $r \leqslant 3$. The equivalence between (3) and (4) follows from Property (i) of Theorem 5.1. The equivalence between (4) and (5) follows from the definitions. □

Proof of Theorem 1.9. Since y satisfies $(\mathscr{H}-\mathcal{O}_r)$ with $r \in \{1,2,3\}$, Proposition 6.2 implies that y also satisfies $(\mathscr{P}-\mathcal{O}_r)$. Then, by [FR25, Theorem 9] the minimal p-Mahler equation of y over $\mathbb{K}_\infty = \overline{\mathbb{Q}}(z^{\frac{1}{\star}})$ has a basis of solutions in $\mathcal{R}$, satisfying $(\mathscr{P}-\mathcal{O}_r)$. Applying Proposition 6.2 again, we conclude that these solutions all satisfy $(\mathscr{H}-\mathcal{O}_r)$, as wanted. □

7. p-REGULAR ELEMENTS OF $\mathbb{K}^{\mathbb{Q}}$

Let $\mathbb{K}$ be a field. In a landmark paper, Allouche and Shallit [AS92] introduced the notion of p-regular sequences as a generalization of the notion of p-automatic sequences. We recall that a sequence $(a_n)_{n\in\mathbb{Z}_{\geqslant 0}} \in \mathbb{K}^{\mathbb{Z}_{\geqslant 0}}$ is p-regular if its p-kernel, given by

$$\left\{(a_{p^i n+j})_{n\in\mathbb{Z}_{\geqslant 0}} \mid i \in \mathbb{Z}_{\geqslant 0},\, j \in \{0,\ldots,p^i-1\}\right\} \subset \mathbb{K}^{\mathbb{Z}_{\geqslant 0}},$$

spans a finite dimensional $\mathbb{K}$-vector space. These sequences admit many other characterizations. For instance, they are:

- the sequences produced by linear representations, that is, the sequences $(a_n)_{n\in\mathbb{Z}_{\geqslant 0}} \in \mathbb{K}^{\mathbb{Z}_{\geqslant 0}}$ with general term
$$a_n = \boldsymbol{\tau} A_{i_t} \cdots A_{i_0} \boldsymbol{\lambda}$$
where $\boldsymbol{\tau} \in \mathbb{K}^{1\times m}$, $A_0,\ldots,A_{p-1} \in \mathbb{K}^{m\times m}$, $\boldsymbol{\lambda} \in \mathbb{K}^{m\times 1}$ for some $m \geqslant 0$ and where $i_t \ldots i_0 = (n)_p$ is the base-p expansion of the integer n [AS92, Lemma 4.1];

- the sequences $(a_n)_{n\in\mathbb{Z}_{\geqslant 0}} \in \mathbb{K}^{\mathbb{Z}_{\geqslant 0}}$ such that the series $\sum_{n\geqslant 0} a_n (n)_p$ in the non-commutative indeterminates over the alphabet $\{0,\ldots,p-1\}$ is a rational series (see [AS92, Theorem 4.3] and Section 7.1);
- the sequences $(a_n)_{n\in\mathbb{Z}_{\geqslant 0}} \in \mathbb{K}^{\mathbb{Z}_{\geqslant 0}}$ produced by finite weighted automata [BR88, Chapter I, Proposition 6.1].

We say that a power series $\sum_{n\geqslant 0} a_n z^n \in \mathbb{K}[[z]]$ is p-regular if the sequence $(a_n)_{n\in\mathbb{Z}_{\geqslant 0}}$ is p-regular.

The aim of the present section is to extend the notion of p-regular sequence to the elements of $\mathbb{K}^{\mathbb{Q}}$. This will be used in Section 8 to extend the notion of p-regular power series to Hahn series. The notion of p-kernel does not extend directly to elements of $\mathbb{K}^{\mathbb{Q}}$, so we instead resort to the characterization in terms of rational series.

7.1. **Rational series.** Let Σ be a finite alphabet, that is a finite set of symbols. Let $\Sigma^\star$ denote the set of finite words over Σ, that is, the free monoid spanned by Σ. An element of $\Sigma^\star$ is either the empty word ϵ or an expression of form $s_1\cdots s_t$ for some $t\in\mathbb{Z}_{\geqslant 1}$ and some $s_1,\ldots,s_t\in\Sigma$. Let $\mathbb{K}\langle\langle\Sigma\rangle\rangle$ denote the $\mathbb{K}$-algebra of non-commutative formal series in the non-commutative indeterminates $s\in\Sigma$. We denote by $\mathbb{K}\langle\Sigma\rangle$ the sub-$\mathbb{K}$-algebra of $\mathbb{K}\langle\langle\Sigma\rangle\rangle$ made of the non-commutative polynomials in the non-commutative indeterminates $s\in\Sigma$. An element of $\mathbb{K}\langle\langle\Sigma\rangle\rangle$ can be written as a formal sum of the form

$$a = \sum_{w\in\Sigma^\star} a_w w$$

with $a_w \in \mathbb{K}$. Such an element of $\mathbb{K}\langle\langle\Sigma\rangle\rangle$ belongs to $\mathbb{K}\langle\Sigma\rangle$ if and only if $a_w = 0$ for all but finitely many $w\in\Sigma^\star$. We denote by $\mathbb{K}\langle\langle\Sigma\rangle\rangle_{rat}$ the set of rational series; this is by definition the smallest subset of $\mathbb{K}\langle\langle\Sigma\rangle\rangle$ that contains $\mathbb{K}\langle\Sigma\rangle$ and is closed under addition, multiplication and the operation

$$a \mapsto a^* := (1-a)^{-1} = \sum_{n\geqslant 0} a^n$$

whenever a has zero constant coefficient, *i.e.*, $a_\epsilon = 0$.

For the proof of the following result, we refer to [BR88]. Recall that the Hadamard product $a\odot b \in \mathbb{K}\langle\langle\Sigma\rangle\rangle$ of two formal series $a = \sum_{w\in\Sigma^\star} a_w w \in \mathbb{K}\langle\langle\Sigma\rangle\rangle$ and $b = \sum_{w\in\Sigma^\star} b_w w \in \mathbb{K}\langle\langle\Sigma\rangle\rangle$ is defined by

$$a\odot b = \sum_{w\in\Sigma^\star} a_w b_w w.$$

Proposition 7.1. *The set $\mathbb{K}\langle\langle\Sigma\rangle\rangle_{rat}$ of rational series is a sub-$\mathbb{K}$-algebra of $\mathbb{K}\langle\langle\Sigma\rangle\rangle$ containing $\mathbb{K}\langle\Sigma\rangle$ and closed under the Hadamard product.*

The following result follows from Schützenberger Theorem; see for instance [BR88, Chapter 1, Theorem 7.1].

Proposition 7.2. *A formal series $a = \sum_{w\in\Sigma^\star} a_w w \in \mathbb{K}\langle\langle\Sigma\rangle\rangle$ is rational if and only if there exist $m\in\mathbb{Z}_{\geqslant 0}$, a morphism of monoid $\mu : \Sigma^\star \to \mathbb{K}^{m\times m}$, a row vector $\boldsymbol{\tau}\in\mathbb{K}^{1\times m}$ and a column vector $\boldsymbol{\lambda}\in\mathbb{K}^{m\times 1}$ such that, for any $w\in\Sigma^\star$,*

$$a_w = \boldsymbol{\tau}\mu(w)\boldsymbol{\lambda}\,.$$

Such a triplet $(\mu,\boldsymbol{\tau},\boldsymbol{\lambda})$ is called a linear representation of rank m of a.

Definition 7.3. *A language over* Σ *is a subset* L *of* $\Sigma^\star$. *We say that a language* L *over* Σ *is regular if its characteristic series* $\sum_{w\in L} w$ *is rational.*

Regular languages over a finite alphabet are sometimes also called *recognizable languages* or *rational languages*. The following lemmas will be used later.

Lemma 7.4. *Let* $a = \sum_{w\in\Sigma^\star} a_w w \in \mathbb{K}\langle\langle\Sigma\rangle\rangle_{rat}$ *be a rational series and let* $L \subset \Sigma^\star$ *be a regular language. Then, the formal series* $\sum_{w\in L} a_w w$ *is rational.*

Proof. The characteristic series $b = \sum_{w\in L} w$ of L is rational because L is a regular language. Since a is a rational series, it follows from Proposition 7.1 that the series

$$a \odot b = \sum_{w\in L} a_w w$$

is rational. $\square$

When taking a to be the characteristic series of a language in the previous lemma, we immediately obtain the following result.

Corollary 7.5. *The intersection of two regular languages over* Σ *is a regular language as well.*

7.2. p**-Regular elements of** $\mathbb{K}^{\mathbb{Q}}$**.** Let $\blacksquare$ be a symbol and consider the following alphabet:

$$\Sigma_{\blacksquare,p} = \{0,\ldots,p-1,\blacksquare\}.$$

We let L_p denote the language over $\Sigma_{\blacksquare,p}$ composed of the words of the form

$$s_1\cdots s_t \blacksquare s_{t+1}\cdots s_{t+u}$$

with $t,u \in \mathbb{Z}_{\geqslant 0}$ and $s_i \in \{0,\ldots,p-1\}$, with $s_1 \neq 0$ if $t \geqslant 1$ and $s_{t+u} \neq 0$ if $u \geqslant 1$. For $t = 0$ (resp. $u = 0$), $s_1\cdots s_t$ (resp. $s_{t+1}\cdots s_{t+u}$) denotes the empty word ϵ. In particular, setting $t = u = 0$, we obtain that $\blacksquare$ belongs to L_p.

Proposition 7.6. *The language* L_p *over* $\Sigma_{\blacksquare,p}$ *is regular.*

Proof. The language $L_\blacksquare$ made of all the words over $\Sigma_{\blacksquare,p}$ containing exactly one symbol $\blacksquare$ is regular. Indeed, it is easily seen that a linear representation for its characteristic series is given by the triplet $(\mu,\boldsymbol{\tau},\boldsymbol{\lambda})$ where $\mu : (\Sigma_{\blacksquare,p})^\star \to \mathbb{Q}^{2\times 2}$ is the unique morphism of monoid such that

$$\mu(\blacksquare) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and, for all } s \in \{0,\ldots,p-1\},\quad \mu(s) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and where

$$\boldsymbol{\tau} = (1,1),\ \boldsymbol{\lambda} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Similarly, it is easily seen that the language made of the words over $\Sigma_{\blacksquare,p}$ not starting with a 0 and the language of words not ending with a 0 are regular. The language L_p being the intersection of these three regular languages, it is regular as well thanks to Corollary 7.5. $\square$

The words of L_p are in one-to-one correspondence with the elements $\mathbb{Z}[p^{-1}]_{\geqslant 0}$ via the bijection

$$\begin{array}{rcl} \|\cdot\| : L_p & \to & \mathbb{Z}[p^{-1}]_{\geqslant 0} \\ s_1 \cdots s_t \blacksquare s_{t+1} \cdots s_{t+u} & \mapsto & \|s_1 \cdots s_t \blacksquare s_{t+1} \cdots s_{t+u}\| = \sum_{i=1}^{t} s_i p^{t-i} + \sum_{i=1}^{u} s_{t+i} p^{-i}. \end{array}$$

We denote the reciprocal function by

$$[\,\cdot\,]_p : \mathbb{Z}[p^{-1}]_{\geqslant 0} \to L_p.$$

We view $[\gamma]_p$ as our preferred base-p representation of $\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}$.

Example 7.7. We have:

$$\begin{array}{rcl} [0]_p & = & \blacksquare \\ [1]_p & = & 1\blacksquare \\ [1/p]_p & = & \blacksquare 1 \\ [p]_p & = & 10\blacksquare \end{array} \qquad\qquad \begin{array}{rcl} \|\blacksquare\| & = & 0 \\ \|1\blacksquare\| & = & 1 \\ \|\blacksquare 1\| & = & 1/p \\ \|10\blacksquare\| & = & p \end{array}$$

Definition 7.8. *We say that $(a_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{K}^{\mathbb{Q}}$ is p-regular if its support is included in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$ and if the formal series*

$$\sum_{\gamma\in\mathbb{Z}[p^{-1}]_{\geqslant 0}} a_\gamma [\gamma]_p \in \mathbb{K}\langle\langle \Sigma_{\blacksquare,p} \rangle\rangle$$

is rational.

Here are several other characterizations of p-regular elements of $\mathbb{K}^{\mathbb{Q}}$.

Proposition 7.9. *Let $(a_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{K}^{\mathbb{Q}}$ with support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. The following are equivalent:*

(a) *$(a_\gamma)_{\gamma\in\mathbb{Q}}$ is p-regular;*

(b) *we have*

$$\sum_{\gamma\in\mathbb{Z}[p^{-1}]_{\geqslant 0}} a_\gamma [\gamma]_p \in \mathbb{K}\langle\langle \Sigma_{\blacksquare,p} \rangle\rangle_{rat};$$

(c) *there exists*

$$\sum_{w\in(\Sigma_{\blacksquare,p})^\star} b_w w \in \mathbb{K}\langle\langle \Sigma_{\blacksquare,p} \rangle\rangle_{rat}$$

such that, for all $w \in L_p$, we have $b_w = a_{\|w\|}$;

(d) *there exists a linear representation $(\mu : (\Sigma_{\blacksquare,p})^\star \to \mathbb{K}^{m\times m}, \boldsymbol{\tau}, \boldsymbol{\lambda})$ such that,*

$$\boldsymbol{\tau}\mu(w)\boldsymbol{\lambda} = \begin{cases} a_{\|w\|} & \text{if } w \in L_p, \\ 0 & \text{if } w \in (\Sigma_{\blacksquare,p})^\star \backslash L_p; \end{cases}$$

(e) *there exists a linear representation $(\mu : (\Sigma_{\blacksquare,p})^\star \to \mathbb{K}^{m\times m}, \boldsymbol{\tau}, \boldsymbol{\lambda})$ such that, for all $w \in L_p$, we have*

$$a_{\|w\|} = \boldsymbol{\tau}\mu(w)\boldsymbol{\lambda}.$$

Proof. The equivalence between (a) and (b) is the definition of p-regularity. The fact that (b) implies (c) and the fact that (d) implies (e) are obvious. The equivalence between (b) and (d) and the equivalence between (c) and (e) both follow from Proposition 7.2. That (c) implies (b) is a consequence of Lemma 7.4 and of the fact, established in Proposition 7.6, that L_p is a regular language. ☐

Definition 7.10. *A linear representation* $(\mu : (\Sigma_{\blacksquare,p})^\star \to \mathbb{K}^{m\times m}, \boldsymbol{\tau}, \boldsymbol{\lambda})$ *as in* (d) *of Proposition* 7.9 *will be called a linear representation (of rank m) associated with* $(a_\gamma)_{\gamma\in\mathbb{Q}}$.

Proposition 7.11. *The set of p-regular elements of $\mathbb{K}^{\mathbb{Q}}$ is a sub-$\mathbb{K}$-algebra of $\mathbb{K}^{\mathbb{Q}}$, with respect to the termwise product.*

Proof. This follows immediately from the following three assertions:

- the set of elements of $\mathbb{K}^{\mathbb{Q}}$ with support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$ is a $\mathbb{K}$-algebra;
- $\mathbb{K}\langle\langle\Sigma_{\blacksquare,p}\rangle\rangle_{rat}$ is a $\mathbb{K}$-vector space;
- $\mathbb{K}\langle\langle\Sigma_{\blacksquare,p}\rangle\rangle_{rat}$ is closed under the Hadamard product.

The first assertion is straightforward, while the last two are guaranteed by Proposition 7.1. ☐

We now state two properties which will be of fundamental importance when we will consider the generating series of p-regular elements of $\mathbb{K}^{\mathbb{Q}}$. The first one will allow us to consider truncations while the second one will allow one to perform some gauge change.

Let $(a_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{K}^{\mathbb{Q}}$ be p-regular and let $\delta \in \mathbb{Q}$. We set

$$a_\gamma^{<\delta} = \begin{cases} a_\gamma & \text{if } \gamma < \delta, \\ 0 & \text{else.} \end{cases}$$

We define $(a_\gamma^{>\delta})_{\gamma\in\mathbb{Q}}$, $(a_\gamma^{\leqslant\delta})_{\gamma\in\mathbb{Q}}$ and $(a_\gamma^{\geqslant\delta})_{\gamma\in\mathbb{Q}}$ similarly.

Lemma 7.12. *Let $(a_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{K}^{\mathbb{Q}}$ be p-regular and let $\delta \in \mathbb{Q}$. Then, the sequences $(a_\gamma^{<\delta})_{\gamma\in\mathbb{Q}}$, $(a_\gamma^{>\delta})_{\gamma\in\mathbb{Q}}$, $(a_\gamma^{\leqslant\delta})_{\gamma\in\mathbb{Q}}$ and $(a_\gamma^{\geqslant\delta})_{\gamma\in\mathbb{Q}}$ are p-regular.*

Proof. We prove the result for the sequence $(a_\gamma^{<\delta})_{\gamma\in\mathbb{Q}}$, the other cases being similar. It follows from [Ked17, Example 4.2] that the language

$$L_p^{<\delta} := \{[\gamma]_p \mid \gamma < \delta\}$$

is regular. Since the characteristic series of $(a_\gamma^{<\delta})_{\gamma\in\mathbb{Q}}$ is the restriction to $L_p^{<\delta}$ of the characteristic series $(a_\gamma)_{\gamma\in\mathbb{Q}}$ and since the latter is rational, it follows from Lemma 7.4 that the former is rational. Thus $(a_\gamma^{<\delta})_{\gamma\in\mathbb{Q}}$ is p-regular. ☐

Lemma 7.13. *Let $\nu \in \mathbb{Z}_{\geqslant 0}$, $d \in \mathbb{Z}_{\geqslant 1}$ and $(a_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{K}^{\mathbb{Q}}$. We consider $(b_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{K}^{\mathbb{Q}}$ defined by*

$$b_\gamma = \begin{cases} a_{(\gamma-\nu)/d} & \textit{if } \gamma \in d\mathbb{Z}[p^{-1}]_{\geqslant 0} + \nu, \\ 0 & \textit{else.} \end{cases} \tag{31}$$

Then, the following properties are equivalent:

- $(a_\gamma)_{\gamma\in\mathbb{Q}}$ *is p-regular;*
- $(b_\gamma)_{\gamma\in\mathbb{Q}}$ *is p-regular.*

The proof of Lemma 7.13, given at the end of the current subsection, relies on the fact that the map

$$\begin{array}{ccc} L_p & \to & L_p \\ [\gamma]_p & \mapsto & [d\gamma+\nu]_p \end{array}$$

can be performed by finite state transducers. This fact is classical when $\gamma \in \mathbb{Z}_{\geqslant 0}$; see [AS03, p. 142-143]. When $\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}$, it is stated in [Ked17, Example 4.3]. For the sake of completeness and clarity, we provide a full proof here, after a brief discussion of transducers.

A *finite-state transducer* (FST) is a sextuple

$$\pi = (Q, \Sigma, \delta, q_0, \Delta, \lambda)$$

where

- Q is a finite set whose elements are called states;
- $q_0 \in Q$ is called the initial state;
- Σ is a finite alphabet, called the input alphabet;
- $\delta : Q \times \Sigma \to Q$ is a map called the transition function;
- Δ is a finite alphabet called the output alphabet;
- $\lambda : Q \times \Sigma \to \Delta^\star$ is a map called the output function.

See [AS03, p. 140] for further details. The transition function δ has two extensions

$$\delta^\star_{\text{left}} : Q \times \Sigma^\star \to Q \quad \text{and} \quad \delta^\star_{\text{right}} : Q \times \Sigma^\star \to Q$$

which are recursively defined, for any $q \in Q$, $a \in \Sigma$ and $w \in \Sigma^\star$, by

$$\delta^\star_{\text{left}}(q, \epsilon) = q \quad \text{and} \quad \delta^\star_{\text{left}}(q, aw) = \delta^\star_{\text{left}}(\delta(q, a), w) \tag{32}$$

and

$$\delta^\star_{\text{right}}(q, \epsilon) = q \quad \text{and} \quad \delta^\star_{\text{right}}(q, wa) = \delta^\star_{\text{right}}(\delta(q, a), w).$$

The FST π induces two maps

$$\pi_{\text{left}} : \Sigma^\star \to \Delta^\star \quad \text{and} \quad \pi_{\text{right}} : \Sigma^\star \to \Delta^\star$$

which transform any element of $\Sigma^\star$ into an element of $\Delta^\star$, one when reading from left to right, the other one when reading in the other direction. They are defined, for any $s_1, \ldots, s_t \in \Sigma$, by

$$\begin{aligned}\pi_{\text{left}}(s_1 \cdots s_t) = \lambda(q_0, s_1)\lambda(\delta^\star_{\text{left}}(q_0, s_1), s_2)\lambda(\delta^\star_{\text{left}}(q_0, s_1 s_2), s_3) \cdots \\ \cdots \lambda(\delta^\star_{\text{left}}(q_0, s_1 s_2 \cdots s_{t-1}), s_t)\end{aligned}$$

and

$$\begin{aligned}\pi_{\text{right}}(s_1 \cdots s_t) = \lambda(q_0, s_t)\lambda(\delta^\star_{\text{right}}(q_0, s_t), s_{t-1})\lambda(\delta^\star_{\text{right}}(q_0, s_{t-1} s_t), s_{t-2}) \cdots \\ \cdots \lambda(\delta^\star_{\text{right}}(q_0, s_2 \cdots s_{t-1} s_t), s_1).\end{aligned}$$

Since it reads the word $s_1 \cdots s_t$ from left to right, by an abuse of notation, we call π_{left} a *left* FST. Similarly, we call π_{right} a *right* FST. We say that a left or a right FST is *faithful* if the pre-image of any word in $\Delta^\star$ is finite.

Example 7.14. Let Σ be a finite alphabet and $a \in \Sigma$. We leave to the reader to check that the map $w \mapsto aw$ is a faithful left FST from $\Sigma^\star$ to itself and that the map $w \mapsto wa$ is a faithful right FST from $\Sigma^\star$ to itself.

Example 7.15. Consider a finite alphabet Σ and choose $a \in \Sigma$. Consider the FST $(\{q_0, q_1\}, \Sigma, \delta, q_0, \Sigma, \lambda)$ with $q_0 \neq q_1$ where δ sends any element of $Q \times \Sigma$ to q_1 and where:

$$\lambda(q_0, a) = \epsilon \ \text{ and}, \forall (q, s) \in Q \times \Sigma \backslash \{(q_0, a)\}, \ \lambda(q, s) = s.$$

The induced left (*resp.* right) FST removes the first a on the left (*resp.* on the right) of a word of $\Sigma^\star$ if it begins with an a. It is faithful.

Example 7.16 (Addition)**.** We build a right FST which performs the addition by one on any $\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}$ when reading the word $0[\gamma]_p$. Let

- $Q = \{\bullet, 0, 1\}$ (the symbol $\bullet$ will be used to signal that we haven't read the symbol $\blacksquare$ yet, while the numbers $0, 1$ will be the possible carried numbers when computing the addition);
- $q_0 = \bullet$ be the initial state;
- $\delta : Q \times \Sigma_{\blacksquare,p} \to Q$ be defined as follows:
  - $\delta(\bullet, s) = \bullet$, for all $s \in \{0, \dots, p-1\}$;
  - $\delta(q, \blacksquare) = 1$ for all $q \in Q$;
  - for all $q \in \{0, 1\}$ and all $s \in \{0, \dots, p-1\}$,
$$\delta(q, s) = \lfloor (q+s)/p \rfloor ;$$
- $\lambda : Q \times \Sigma_{\blacksquare,p} \to (\Sigma_{\blacksquare,p})^\star$ be such that
  - for all $s \in \Sigma_{\blacksquare,p}$, $\lambda(\bullet, s) = s$;
  - for all $q \in \{0, 1\}$ and all $s \in \{0, \dots, p-1\}$,
$$\lambda(q, s) = q + s \mod p \in \{0, \dots, p-1\}.$$

Set $\pi = (Q, \Sigma_{\blacksquare,p}, \delta, q_0, \Sigma_{\blacksquare,p}, \lambda)$. We leave to the reader to check that the associated right FST is faithful and that, for any $\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}$,

$$\pi_{\text{right}}(0[\gamma]_p) = 0^u[\gamma + 1]_p \text{ for some } u \in \{0, 1\}. \tag{33}$$

Example 7.17 (Multiplication)**.** We build a faithful right FST to perform the multiplication by an integer $d \in \mathbb{Z}_{\geqslant 1}$. Let

- $Q = \{0, \dots, d-1\}$ (which we think of as the set of possible carried numbers);
- $q_0 = 0$ be the initial state;
- $\delta : Q \times \Sigma_{\blacksquare,p} \to Q$ be defined, for all $q \in Q$, by $\delta(q, \blacksquare) = q$ and
$$\delta(q, s) = \left\lfloor \frac{ds + q}{p} \right\rfloor, \quad \forall s \in \{0, \dots, p-1\};$$
- $\lambda : Q \times \Sigma_{\blacksquare,p} \to Q$ be defined, for all $q \in Q$, by $\lambda(q, \blacksquare) = \blacksquare$ and
$$\lambda(q, s) = ds + q \mod p \in \{0, \dots, p-1\}, \quad \forall s \in \{0, \dots, p-1\}.$$

Set $\pi = (Q, \Sigma_{\blacksquare,p}, \delta, q_0, \Sigma_{\blacksquare,p}, \lambda)$ and let ℓ be such that $p^\ell \geqslant d$. One may check that π_{right} is a faithful right FST and that, for any $\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}$,

$$\pi_{\text{right}}(0^\ell[\gamma]_p) = 0^t[d\gamma]_p 0^u, \qquad \text{for some } t, u \text{ with } 0 \leqslant t, u \leqslant \ell. \tag{34}$$

Lemma 7.13 will follow from the fact that the map $\gamma \mapsto d\gamma + \nu$ can be performed by a composition of some FST and from the following result.

Proposition 7.18. *Let* $a = \sum_{w \in \Sigma^\star} a_w w \in \mathbb{K}\langle\langle \Sigma \rangle\rangle$ *be a formal series and let* π *be a faithful left or right FST. Then, the formal series*

$$\pi(a) = \sum_{w \in \Sigma^\star} a_w \pi(w) \in \mathbb{K}\langle\langle \Delta \rangle\rangle$$

is well-defined. Furthermore, $\pi(a)$ *is rational, if and only if the series* a *is rational.*

Proof. The fact that the formal series $\pi(a)$ is a well-defined element of $\mathbb{K}\langle\langle\Delta\rangle\rangle$ follows directly from the fact that π is faithful. Faithful left or right FST are particular cases of what is called *regulated rational transductions* in [KS86, Chapter 9] and even *polynomial transductions*. The fact that $\pi(a)$ is a rational series if a is a rational series follows from [KS86, Theorem 9.6]. The converse statement follows from [KS86, Theorem 9.13]. ☐

We are now in a position to prove Lemma 7.13.

Proof of Lemma 7.13. For any $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$, we consider the map

$$\begin{array}{rcl}\pi_{d,\nu} : L_p & \to & L_p \\ [\gamma]_p & \mapsto & [d\gamma+\nu]_p.\end{array}$$

Set $a = \sum_{\gamma\in\mathbb{Z}[p^{-1}]_{\geqslant 0}} a_\gamma[\gamma]_p$ and $b = \sum_{\gamma\in\mathbb{Z}[p^{-1}]_{\geqslant 0}} b_\gamma[\gamma]_p$. We have

$$\begin{aligned}\pi_{d,\nu}(a) = \sum_{\gamma\in\mathbb{Z}[p^{-1}]_{\geqslant 0}} a_\gamma[d\gamma+\nu]_p &= \sum_{\gamma'\in d\mathbb{Z}[p^{-1}]_{\geqslant 0}+\nu} a_{(\gamma'-\nu)/d}[\gamma']_p \\ &= \sum_{\gamma'\in\mathbb{Z}[p^{-1}]_{\geqslant 0}} b_{\gamma'}[\gamma']_p = b\,.\end{aligned}$$

So, in view of Proposition 7.18, in order to conclude the proof it is sufficient to prove that $\pi_{d,\nu}$ can be obtained as the composition of some faithful left and right FSTs. Let us prove this is indeed the case. Since $\pi_{d,\nu} = \pi_{1,\nu}\circ\pi_{d,0}$, it is sufficient to prove that $\pi_{1,\nu}$ and $\pi_{d,0}$ are compositions of some faithful FSTs.

Let us first prove that $\pi_{1,\nu}$ is a composition of some faithful FST. Since

$$\pi_{1,\nu} = \underbrace{\pi_{1,1}\circ\cdots\circ\pi_{1,1}}_{\nu \text{ times}},$$

it is sufficient to prove that $\pi_{1,1}$ is the composition of some faithful FST. Consider the faithful left FST $\psi_1 : (\Sigma_{\blacksquare,p})^\star \to (\Sigma_{\blacksquare,p})^\star$ which adds a 0 at the beginning of a word (see Example 7.14) and $\psi_2 : (\Sigma_{\blacksquare,p})^\star \to (\Sigma_{\blacksquare,p})^\star$ which removes the first 0 on the left of a word if it begins with a 0 (see Example 7.15). Consider the faithful right FST ψ_3 built in Example 7.16. Then, it follows from (33) that $\pi_{1,1} = \psi_2\circ\psi_3\circ\psi_1$. In particular, $\pi_{1,1}$ is a composition of some faithful FST.

Let us now prove that $\pi_{d,0}$ is a composition of some faithful FST. Let ψ_4 be the faithful right FST built in Example 7.17 and let ℓ be the least integer such that $p^\ell \geqslant d$. Let ψ_5 denote the faithful right FST which removes the first 0 on the right of an element of $\Sigma_{\blacksquare,p}$, if it starts with a 0 (see Example 7.15). Then, it follows from (34) that $\pi_{d,0} = \psi_5^\ell\circ\psi_2^\ell\circ\psi_4\circ\psi_1^\ell$ and, hence, $\pi_{d,0}$ is the composition of some faithful FST. ☐

7.3. **Regular sequences in the classical sense.** As mentioned at the very beginning of Section 7, Allouche and Shallit [AS92] defined a notion of p-regularity for the elements of $\mathbb{K}^{\mathbb{Z}_{\geqslant 0}}$. In this section, we prove that an element $(a_\gamma)_{\gamma\in\mathbb{Q}}$ of $\mathbb{K}^{\mathbb{Q}}$ with support in $\mathbb{Z}_{\geqslant 0}$ is p-regular in our sense if and only the sequence $(a_n)_{n\geqslant 0} \in \mathbb{K}^{\mathbb{Z}_{\geqslant 0}}$ is p-regular in the sense of [AS92].

Let $\Sigma_p = \{0,\ldots,p-1\}$. Set $(0)_p = \epsilon$ and, for $n \in \mathbb{Z}_{\geqslant 1}$,

$$(n)_p = s_t\cdots s_0 \in \Sigma_p^\star \text{ with } s_t \neq 0 \text{ and } n = s_0 + s_1 p + \cdots + s_t p^t.$$

A sequence $(a_n)_{n\in\mathbb{Z}_{\geqslant 0}} \in \mathbb{K}^{\mathbb{Z}_{\geqslant 0}}$ is p-regular in the sense of [AS92] if and only if

$$\sum_{n\in\mathbb{Z}_{\geqslant 0}} a_n (n)_p \in \mathbb{K}\langle\langle \Sigma_{\blacksquare,p}\rangle\rangle \tag{35}$$

is rational [AS92, Theorem 4.3]. But, for any $n \in \mathbb{Z}_{\geqslant 0}$, $[n]_p = (n)_p\blacksquare$ and, according to Example 7.14, the map $w \mapsto w\blacksquare$ from $\Sigma^{\star}_{\blacksquare,p}$ to itself can be performed by a faithful right FST. So, it follows from Proposition 7.18 that the series (35) is rational if and only if the series $\sum_{\gamma\in\mathbb{Q}} a_\gamma[\gamma]_p$ is. Thus, the sequence $(a_n)_{n\geqslant 0}$ is p-regular in the sense of [AS92] if and only if its unique extension $(a_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{K}^{\mathbb{Q}}$ with support in $\mathbb{Z}_{\geqslant 0}$ is p-regular in the sense of Definition 7.8.

7.4. **Deterministic finite automata.** A deterministic finite automaton with output (DFAO) is a sextuple $M = (Q, \Sigma, \delta, q_0, \Delta, \tau)$, where

- Q is a finite set called the set of states;
- Σ is a finite alphabet called the input alphabet;
- $\delta : Q \times \Sigma \to Q$ is a map called the transition function;
- $q_0 \in Q$ is called the initial state;
- Δ is a set, called the output set;
- $\tau : Q \to \Delta$ is a map called the output function.

We extend δ to a function $\delta^{\star}_{\text{left}} : Q \times \Sigma^{\star} \to Q$ as in (32). Any DFAO M gives rise to a function

$$\begin{array}{rcl} g_M : \Sigma^{\star} & \to & \Delta \\ w & \mapsto & \tau(\delta^{\star}_{\text{left}}(q_0, w)). \end{array}$$

Remark 7.19. *Note that we could define another function by reading the words from right to left. However, it is a classical result* [AS03, Theorem 5.2.3] *that this does not extend the class of functions we obtain.*

Remark 7.20. *It follows from Kleene's theorem* [AS03, Theorem 4.1.5] *that the regular languages are the languages L for which there exists a DFAO M, with output set $\Delta = \{0, 1\}$ such that $L = g_M^{-1}(1)$.*

The following result states that the series produced by a DFAO are the rational series whose coefficients belong to a finite set.

Proposition 7.21. *Let $a = \sum_{w\in\Sigma^{\star}} a_w w \in \mathbb{K}\langle\langle\Sigma\rangle\rangle$. The following statements are equivalent:*

(1) the series a is rational and $\{a_w \mid w \in \Sigma^{\star}\}$ is a finite set;
(2) there exists a DFAO M such that, for all $w \in \Sigma^{\star}$, $g_M(w) = a_w$.

Proof. Suppose that a is rational and that the set $\mathcal{E} = \{a_w \mid w \in \Sigma^{\star}\}$ is finite. It follows from [BR88, Chapter III, Theorem 2.8] that, for any $e \in \mathcal{E}$, the language $L_e = \{w \in \Sigma^{\star} \mid a_w = e\}$ is regular. Since $(L_e)_{e\in\mathcal{E}}$ is a partition of $\Sigma^{\star}$, it follows from [AS03, Th. 4.3.2] that there exists a DFAO M such that, for all $w \in \Sigma^{\star}$, $g_M(w) = a_w$.

Reciprocally, suppose that there exists a DFAO M such that, for all $w \in \Sigma^{\star}$, $g_M(w) = a_w$. Then, the set $\mathcal{E} = \{g_M(w) \mid w \in \Sigma^{\star}\}$ is finite. For all $e \in \mathcal{E}$, $L_e = \{w \in \Sigma^{\star} \mid g_M(w) = e\}$ is a regular language by Remark 7.20

and, hence, the series $\sum_{w\in L_e} w$ is rational. Since the set of rational series is a $\mathbb{K}$-algebra by Proposition 7.1, we have

$$\sum_{w\in\Sigma^\star} a_w w = \sum_{w\in\Sigma^\star} g_M(w)w = \sum_{e\in\mathscr{E}}\sum_{w\in L_e} ew = \sum_{e\in\mathscr{E}} e\left(\sum_{w\in L_e} w\right) \in \mathbb{K}\langle\langle\Sigma\rangle\rangle_{\mathrm{rat}}.$$

□

7.5. **Other characterizations of p-regularity.** There are several other possible characterizations of p-regular elements of $\mathbb{K}^{\mathbb{Q}}$.

Kedlaya considered in [Ked17] the elements $(a_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{K}^{\mathbb{Q}}$ with support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$ for which there exists an integer m, a row vector $\boldsymbol{\tau} \in \mathbb{K}^m$, two morphisms of monoids $\mu, \nu : \{0, \dots, p-1\}^\star \to \mathbb{K}^{m\times m}$ and a column vector $\boldsymbol{\lambda} \in \mathbb{K}^m$ such that, for all $\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}$,

$$a_\gamma = \boldsymbol{\tau}\mu(s_t \cdots s_1)\nu(s_{t+1}\cdots s_{t+u})\boldsymbol{\lambda} \tag{36}$$

where $[\gamma]_p = s_1 \cdots s_t \blacksquare s_{t+1} \cdots s_{t+u}$. Although this is not trivial, one can show that these sequences coincide with the p-regular sequences introduced in Definition 7.8. However, the rank m of a minimal representation in Kedlaya's model may differ from the one of a minimal linear representation in the sense of Definition 7.10. Theorem 8.24, stated at the end of this paper, highlights that Conditions $(\mathscr{O}\Omega_3), (\mathscr{O}\Omega_4)$ and $(\mathscr{O}\Omega_5)$ have characterizations in terms of minimal linear representation. Such a result would be more intricate for representations of the form (36).

Another characterization of rational series $a = \sum_w a_w w \in \mathbb{K}\langle\langle\Sigma\rangle\rangle$ is that the $\mathbb{K}$-vector space

$$\mathscr{K}_{\mathrm{left}}(a) = \mathrm{span}_{\mathbb{K}}\left\{\sum_w a_{xw}w \mid x\in\Sigma^\star\right\}.$$

is finite dimensional, or, equivalently, that the $\mathbb{K}$-vector space

$$\mathscr{K}_{\mathrm{right}}(a) = \mathrm{span}_{\mathbb{K}}\left\{\sum_w a_{wx}w \mid x\in\Sigma^\star\right\}$$

is finite dimensional (see [BR88, Chapter I, Proposition 5.1]). When considering regularity in the sense introduced by Allouche and Shallit, the space $\mathscr{K}_{\mathrm{right}}(a)$ corresponds to the p-kernel as defined at the beginning of this section. Indeed, when dealing with integers, each map $w \mapsto wx$ corresponds to some map of the form $n \mapsto q^i n + j$. However, in the present setting — unlike in [AS92] — neither the maps $w \mapsto wx$ nor the maps $w \mapsto xw$ can be translated into an arithmetic operation on $\mathbb{Z}[p^{-1}]_{\geqslant 0}$.

Eventually, using the language of automata, one could have defined p-regular elements of $\mathbb{K}^{\mathbb{Q}}$ as those produced by some finite weighted automaton over the alphabet $\Sigma_{\blacksquare,p}$ [BR88, Chapter I, Proposition 6.1].

8. p-Regular and quasi-p-regular Hahn series. Proof of Theorem 1.6.

In this section, we introduce the notions of p-regular and quasi-p-regular Hahn series and we prove Theorem 1.6. Specifically, the proof of case (Reg) of Theorem 1.6 is given in Section 8.5, while the proof of case (Aut) of Theorem 1.6 is provided in Section 8.6.

8.1. p-Regular and quasi-p-regular Hahn series.

In [Ked17], Kedlaya introduced the notion of φ-biautomatic Hahn series and φ-quasi-biautomatic Hahn series, following the approach sketched in Section 7.5. This work has served as a source of inspiration for us, and many of the results we present in this section have counterparts in Kedlaya's paper.

Definition 8.1. *We say that a Hahn series* $f(z) = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$ *is* p*-regular if* $(f_\gamma)_{\gamma\in\mathbb{Q}} \in \overline{\mathbb{Q}}^{\mathbb{Q}}$ *is* p*-regular in the sense of Definition 7.8. In this case, a linear representation associated with* $f(z)$ *is a linear representation associated with* $(f_\gamma)_{\gamma\in\mathbb{Q}}$ *in the sense of Definition 7.10.*

Remark 8.2. *If* $f(z) = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$ *is* p*-regular, then* $(f_\gamma)_{\gamma\in\mathbb{Q}}$ *is* p*-regular, but not every* p*-regular sequence arises in this way because the support of a* p*-regular sequence is not necessarily well-ordered.*

By definition, the support of a p-regular Hahn series is a subset of $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. This is too restrictive for our purpose as the support of a p-Mahler Hahn series may not be a subset of $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. However, it turns out that, up to a change of gauge, we can always reduce to this situation. Precisely, it follows immediately from the standard decomposition (7) recalled in Section 2.3 that, for any p-Mahler Hahn series $f(z) \in \mathscr{H}$, there exist $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that the support of $z^\nu f(z^d)$ is included in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. This motivates the following definition.

Definition 8.3. *We say that a Hahn series* $f(z) \in \mathscr{H}$ *is quasi-*p*-regular if there exist* $\nu \in \mathbb{Z}_{\geqslant 0}$ *and* $d \in \mathbb{Z}_{\geqslant 1}$ *such that* $z^\nu f(z^d)$ *is* p*-regular.*[2]

As shown by the following result, an important feature in this definition is that the p-regularity does not depend on the choice of a pair (ν, d) for which $\operatorname{supp} z^\nu f(z^d) \subset \mathbb{Z}[p^{-1}]_{\geqslant 0}$.

Proposition 8.4. *Let* $f(z) \in \mathscr{H}$. *The following properties are equivalent:*

1. $f(z)$ *is quasi-*p*-regular;*
2. *there exist* $\nu \in \mathbb{Z}_{\geqslant 0}$ *and* $d \in \mathbb{Z}_{\geqslant 1}$ *such that* $z^\nu f(z^d)$ *is* p*-regular;*
3. *there exist* $\nu \in \mathbb{Z}_{\geqslant 0}$ *and* $d \in \mathbb{Z}_{\geqslant 1}$ *such that the support of* $z^\nu f(z^d)$ *is included in* $\mathbb{Z}[p^{-1}]_{\geqslant 0}$ *and, for all such* ν, d, *the Hahn series* $z^\nu f(z^d)$ *is* p*-regular.*

The proof appears below, after the following lemma, which is analogous to [Ked17, Lemma 6.4].

Lemma 8.5. *Let* $f(z) \in \mathscr{H}$ *with support in* $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. *Then* $f(z)$ *is* p*-regular if and only if there exist* $\nu \in \mathbb{Z}_{\geqslant 0}$ *and* $d \in \mathbb{Z}_{\geqslant 1}$ *such that the Hahn series* $z^\nu f(z^d)$ *is* p*-regular.*

---

[2] In [AS03], Allouche and Shallit introduce quasi-p-automatic sequences as those that coincide, up to many terms, with an automatic sequence. We draw the reader's attention to the fact that this notion bears no relation to our definition of quasi-p-regular series.

Proof. Consider $f(z) \in \mathscr{H}$ with support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$, $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$. Setting $f(z) = \sum_{\gamma \in \mathbb{Q}} a_\gamma z^\gamma$, we have $z^\nu f(z^d) = \sum_{\gamma \in \mathbb{Q}} b_\gamma z^\gamma$ where $(b_\gamma)_{\gamma \in \mathbb{Q}} \in \overline{\mathbb{Q}}^{\mathbb{Q}}$ is defined by (31). It follows from Lemma 7.13 that $f(z)$ is p-regular if and only if $z^\nu f(z^d)$ is p-regular. □

Proof of Proposition 8.4. The equivalence between (1) and (2) is the definition of quasi-p-regularity. The fact that (3) implies (2) is obvious. Let us prove that (2) implies (3).

Assume that there exist $\mu \in \mathbb{Z}_{\geqslant 0}$ and $c \in \mathbb{Z}_{\geqslant 1}$ such that

$$g(z) := z^\mu f(z^c)$$

is p-regular. In particular, the support of $z^\mu f(z^c)$ is included in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$ and, hence, it only remains to prove that, for all $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that the support of $z^\nu f(z^d)$ is included in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$, the Hahn series $z^\nu f(z^d)$ is p-regular. Consider such ν, d and set

$$h(z) = z^\nu f(z^d).$$

Let $e \in \mathbb{Z}_{\geqslant 1}$ be a common multiple of c and d and set $c' = e/c \in \mathbb{Z}_{\geqslant 1}$, $d' = e/d \in \mathbb{Z}_{\geqslant 1}$ and $\omega = \max\{\mu c', \nu d'\} \in \mathbb{Z}_{\geqslant 0}$. As $g(z)$ is p-regular, it follows from Lemma 8.5 that

$$k(z) := z^{\omega - \mu c'} g(z^{c'})$$

is p-regular as well. Therefore, the equality

$$z^{\omega - \nu d'} h(z^{d'}) = z^{\omega - \nu d'} z^{\nu d'} f(z^{dd'}) = z^\omega f(z^e) = z^{\omega - \mu c'} z^{\mu c'} f(z^{cc'}) = k(z)$$

and Lemma 8.5 imply that $h(z)$ is p-regular. □

Let us note the following immediate consequence of Proposition 8.4.

Corollary 8.6. *Let $f(z), g(z) \in \mathscr{H}$ be quasi-p-regular Hahn series. There exist $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that $z^\nu f(z^d)$ and $z^\nu g(z^d)$ are both p-regular.*

The following result corresponds to [Ked17, Remark 6.6, (a), (b)].

Proposition 8.7. *We have:*

1. *the set of p-regular Hahn series is a $\overline{\mathbb{Q}}$-vector space, invariant by Hadamard product and containing $\overline{\mathbb{Q}}[z]$;*
2. *the set of quasi-p-regular Hahn series is a $\overline{\mathbb{Q}}$-vector space, invariant by Hadamard product and containing $\mathbb{K}_\infty = \overline{\mathbb{Q}}(z^{\frac{1}{\star}})$.*

Proof. The first statement (1) follows immediately from the definitions and from Proposition 7.1.

Let us prove (2). Let $f, g \in \mathscr{H}$ be quasi-p-regular Hahn series. Corollary 8.6 ensures that there exist $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that both $z^\nu f(z^d)$ and $z^\nu g(z^d)$ are p-regular. Therefore, for any $\lambda \in \overline{\mathbb{Q}}$, we have that $z^\nu (f + \lambda g)(z^d) = z^\nu f(z^d) + \lambda z^\nu g(z^d)$ is p-regular by (1), so $f + \lambda g$ is quasi-p-regular. This implies that the set of quasi-p-regular Hahn series is a $\overline{\mathbb{Q}}$-vector space. Moreover, by (1), the Hadamard product of $z^\nu f(z^d)$ and $z^\nu g(z^d)$ is p-regular, but the latter Hadamard product is equal to $z^\nu h(z^d)$ where h is the Hadamard product of f and g, thus h is quasi-p-regular. This justifies the stability of the set of quasi-p-regular Hahn series by Hadamard product. The fact that any element of $\mathbb{K}_\infty$ is quasi-p-regular follows immediately from the fact that any element of $\overline{\mathbb{Q}}[z]$ is p-regular. □

Actually, the set of p-regular Hahn series is a $\overline{\mathbb{Q}}[z]$-algebra and the set of quasi-p-regular Hahn series is a $\mathbb{K}_\infty$-algebras but this is much more difficult to prove. This will follow from Theorem 1.6 (see Corollary 8.20).

The following two lemmas will be used later. They are to compare with [Ked17, Remark 6.6, (e)] and [Ked17, Corollary 5.7] respectively.

Lemma 8.8. *For any $\delta \in \mathbb{Q}$ and any quasi-p-regular $f(z) = \sum_{\gamma\in\mathbb{Q}} a_\gamma z^\gamma \in \mathscr{H}$, the truncations $f_{\leqslant\delta}(z) = \sum_{\gamma\in\mathbb{Q}_{\leqslant\delta}} a_\gamma z^\gamma$, $f_{<\delta}(z) = \sum_{\gamma\in\mathbb{Q}_{<\delta}} a_\gamma z^\gamma$, $f_{\geqslant\delta}(z) = \sum_{\gamma\in\mathbb{Q}_{\geqslant\delta}} a_\gamma z^\gamma$ and $f_{>\delta}(z) = \sum_{\gamma\in\mathbb{Q}_{>\delta}} a_\gamma z^\gamma$ of $f(z)$ are quasi-p-regular.*

Proof. We prove that $f_{<\delta}(z)$ is quasi-p-regular, the proofs in the other cases being similar. Let $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that $g(z) = z^\nu f(z^d)$ has support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. Proposition 8.4 ensures that $g(z)$ is p-regular. Since $g_{<\delta'}(z) = z^\nu f_{<\delta}(z^d)$ where $\delta' = \nu + \delta d$, Proposition 8.4 again shows that, in order to prove that $f_{<\delta}(z)$ is quasi-p-regular, it is sufficient to prove that $g_{<\delta'}(z)$ is p-regular. This is an immediate consequence of Lemma 7.12. ☐

Consider the bijective map

$$\begin{aligned} \mathrm{rev} : (\Sigma_{\blacksquare,p})^\star &\rightarrow (\Sigma_{\blacksquare,p})^\star \\ s_1 \cdots s_t &\mapsto s_t \cdots s_1. \end{aligned}$$

It induces bijections $L_p \to L_p$, $\mathbb{Z}[p^{-1}]_{\geqslant 0} \to \mathbb{Z}[p^{-1}]_{\geqslant 0}$ and $[0,1[\cap\mathbb{Z}[p^{-1}] \to \mathbb{Z}_{\geqslant 0}$, still denoted by rev, making the following diagram commutative:

$$\begin{tikzcd}
{[0,1[\cap\mathbb{Z}[p^{-1}]} \arrow[r, hook] \arrow[d, "\mathrm{rev}"] & \mathbb{Z}[p^{-1}]_{\geqslant 0} \arrow[r, "{[\cdot]_p}"] \arrow[d, "\mathrm{rev}"] & L_p \arrow[r, hook] \arrow[d, "\mathrm{rev}"] & (\Sigma_{\blacksquare,p})^\star \arrow[d, "\mathrm{rev}"] \\
\mathbb{Z}_{\geqslant 0} \arrow[r, hook] & \mathbb{Z}[p^{-1}]_{\geqslant 0} \arrow[r, "{[\cdot]_p}"] & L_p \arrow[r, hook] & (\Sigma_{\blacksquare,p})^\star
\end{tikzcd}$$

For any $f(z) = \sum_{\gamma\in\mathbb{Q}} a_\gamma z^\gamma \in \mathscr{H}$ with support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$, we set

$$f^{\mathrm{rev}}(z) = \sum_{\gamma\in\mathbb{Z}[p^{-1}]_{\geqslant 0}} a_\gamma z^{\mathrm{rev}(\gamma)}.$$

Note that it might not be a Hahn series, for its support is not necessarily well-ordered.

Lemma 8.9. *The following properties relative to a given $f(z) \in \mathscr{H}$ are equivalent:*

(1) *$f(z)$ is p-regular with support in $[0,1[$;*
(2) *$f^{\mathrm{rev}}(z)$ belongs to $\overline{\mathbb{Q}}[[z]]$ and is p-regular.*

Proof. Let us prove (1)⇒(2). Since $f(z)$ is p-regular, its support is included in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. Since it is also included in $[0,1[$, the support of $f(z)$ is actually included in $[0,1[\cap\mathbb{Z}[p^{-1}]_{\geqslant 0}$ and, hence, the support of $f^{\mathrm{rev}}(z)$ is included in $\mathbb{Z}_{\geqslant 0}$, *i.e.*, $f^{\mathrm{rev}}(z)$ belongs to $\overline{\mathbb{Q}}[[z]]$.

Moreover, let $(\mu, \boldsymbol{\tau}, \boldsymbol{\lambda})$ be a linear representation associated with $f(z)$ in the sense of Definition 8.1. Consider the morphism of monoids $\mu^\top : (\Sigma_{\blacksquare,p})^\star \to \overline{\mathbb{Q}}^{m\times m}$ defined by

$$\mu^\top(s_1 \cdots s_t) = \mu(s_1)^\top \cdots \mu(s_t)^\top.$$

Then, $(\mu^\top, \boldsymbol{\lambda}^\top, \boldsymbol{\tau}^\top)$ is a linear representation associated with the power series $f^{\mathrm{rev}}(z)$ in the sense of Definition 8.1. Thus, $f^{\mathrm{rev}}(z)$ is p-regular.

Let us prove (2)⇒(1). Since the support of $f^{\mathrm{rev}}(z)$ is included in $\mathbb{Z}_{\geqslant 0}$, the support of $f(z)$ is included in $[0,1[\cap\mathbb{Z}[p^{-1}]_{\geqslant 0}$. Let $(\mu, \boldsymbol{\tau}, \boldsymbol{\lambda})$ be a linear representation associated with $f^{\mathrm{rev}}(z)$ in the sense of Definition 8.1. Then, $(\mu^\top, \boldsymbol{\lambda}^\top, \boldsymbol{\tau}^\top)$ defined as above is a linear representation associated with $f(z)$ in the sense of Definition 8.1. Thus, $f(z)$ is p-regular. ☐

8.2. **p-Regular power and Puiseux series.** In what follows, we will use results about p-regular power series from [ABS23]. It follows from the discussion in Section 7.3 that the p-regular power series under consideration in [ABS23] are p-regular in the sense of Definition 8.1.

Any p-regular power series is p-Mahler, but the converse implication is false. Theorem 1.2 gives a characterization of the p-regular power series among p-Mahler power series in terms of the arithmetic growth of their coefficients. This can be easily extended to Puiseux series.

Corollary 8.10. *For any p-Mahler Puiseux series $f(z) \in \mathscr{P}$, the following statements are equivalent:*

- (a) *$f(z)$ is quasi-p-regular;*
- (b) *$f(z)$ satisfies $(\mathscr{O}_3)$ or, equivalently, it satisfies $(\mathscr{O}\Omega_3)$, $(\mathscr{O}\Omega_4)$ or $(\mathscr{O}\Omega_5)$.*

Proof. The fact that $(\mathscr{O}_3)$ is equivalent to $(\mathscr{O}\Omega_3)$, $(\mathscr{O}\Omega_4)$ or $(\mathscr{O}\Omega_5)$ for a p-Mahler Puiseux series follows from Proposition 5.2 and from the definitions of $(\mathscr{O}_i)$ and $(\mathscr{O}\Omega_i)$ given in Section 1.

Let $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ be such that $g(z) = z^\nu f(z^d)$ belongs to $\overline{\mathbb{Q}}[[z]]$. From Proposition 8.4, $f(z)$ is quasi-p-regular if and only if $g(z)$ is p-regular. Moreover, according to Lemma 3.2, $f(z)$ satisfies $(\mathscr{O}_3)$ if and only if $g(z)$ satisfies $(\mathscr{O}_3)$ if and only if $g(z)$ satisfies $(\mathscr{O}\Omega_3)$, $(\mathscr{O}\Omega_4)$ or $(\mathscr{O}\Omega_5)$. Now, the result follows from Theorem 1.2. ☐

8.3. **Quasi-p-regularity of the Hahn series $\xi_{\boldsymbol{\omega}}$.** A first step toward the proof of Theorem 1.6 is the following result.

Proposition 8.11. *For all $\boldsymbol{\omega} \in \boldsymbol{\Lambda}$, the Hahn series $\xi_{\boldsymbol{\omega}}$ are quasi-p-regular.*

The proof of Proposition 8.11, given at the end of this section, proceeds by well-founded induction. To this end, we introduce the partial well-order $\prec$ on $\bigcup_{t\geqslant 0}\mathbb{Z}_{\geqslant 0}^t$ defined as follows. For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{Z}_{\geqslant 0}^t$ and $\boldsymbol{\alpha}' = (\alpha'_1, \ldots, \alpha'_{t'}) \in \mathbb{Z}_{\geqslant 0}^{t'}$, we write $\boldsymbol{\alpha}' \prec \boldsymbol{\alpha}$ if one of the following holds:

- $t' < t$,
- $t' = t \geqslant 1$ and $\alpha'_1 < \alpha_1$.

The key ingredient in setting up the well-founded induction is the following result.

Lemma 8.12. *For every $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in \boldsymbol{\Lambda}$, there exists $\lambda \in \overline{\mathbb{Q}}^\times$ and a finite set $E \subset \{\boldsymbol{\omega}' = (\boldsymbol{\alpha}', \boldsymbol{\lambda}', \boldsymbol{a}') \in \boldsymbol{\Lambda} \,|\, \boldsymbol{\alpha}' \prec \boldsymbol{\alpha}\}$ such that*

$$\xi_{\boldsymbol{\omega}}(z^p) - \lambda\xi_{\boldsymbol{\omega}}(z) \in \sum_{\boldsymbol{\omega}'\in E} \overline{\mathbb{Q}}[z^{-\frac{1}{\star}}]\xi_{\boldsymbol{\omega}'}(z)\,. \tag{37}$$

Proof. This is a straightforward consequence of [FR25, Lemma 12]. ☐

We will also use the following result, which is to compare with [Ked17, Lemma 9.1].

Lemma 8.13. *Consider* $f(z), g(z) \in \mathscr{H}$ *and* $\lambda \in \overline{\mathbb{Q}}$ *such that*

$$f(z^p) + \lambda f(z) = g(z).$$

If the support of $f(z)$ *is bounded and if* $g(z)$ *is quasi-*p*-regular, then* $f(z)$ *is quasi-*p*-regular as well.*

Proof. By quasi-p-regularity of $g(z)$, there exists $d \in \mathbb{Z}_{\geqslant 1}$ such that the support of $g(z^d)$ is included in $\mathbb{Z}[p^{-1}]$. Since

$$f((z^d)^p) + \lambda f(z^d) = g(z^d)$$

and since $f(z^d)$ (resp. $g(z^d)$) is quasi-p-regular if and only if $f(z)$ (resp. $g(z)$) is quasi-p-regular by Proposition 8.4, we see that it is sufficient to prove the result in the case when the support of $g(z)$ is included in $\mathbb{Z}[p^{-1}]$.

We assume from now on that the support of $g(z)$ is included in $\mathbb{Z}[p^{-1}]$. We consider the decompositions

$$f(z) = f_{<0}(z) + f_0 + f_{>0}(z) \text{ and } g(z) = g_{<0}(z) + g_0 + g_{>0}(z)$$

with the notations of Lemma 8.8 for the truncations (*i.e.*, the supports of $f_{<0}(z)$ and $g_{<0}(z)$ are included in $\mathbb{Z}[p^{-1}]_{<0}$, the supports of $f_{>0}(z)$ and $g_{>0}(z)$ are included in $\mathbb{Z}[p^{-1}]_{>0}$ and $f_0, g_0 \in \overline{\mathbb{Q}}$). We have

$$\begin{array}{rcl} f_{>0}(z^p) + \lambda f_{>0}(z) & = & g_{>0}(z), \\ f_0 + \lambda f_0 & = & g_0, \\ f_{<0}(z^p) + \lambda f_{<0}(z) & = & g_{<0}(z). \end{array}$$

Note that the support of $f(z)$ is bounded if and only if the support of $f_{>0}(z)$ is bounded. Note also that, by Proposition 8.7 and Lemma 8.8, $f(z)$ (resp. $g(z)$) is quasi-p-regular if and only if both $f_{<0}(z)$ and $f_{>0}(z)$ (resp. $g_{<0}(z)$ and $g_{>0}(z)$) are quasi-p-regular. So, it is sufficient to prove the result in the case when the supports of $f(z)$ and $g(z)$ are included in either $\mathbb{Z}[p^{-1}]_{>0}$ or $\mathbb{Z}[p^{-1}]_{<0}$.

Let us first assume that the supports of $f(z)$ and $g(z)$ are included in $\mathbb{Z}[p^{-1}]_{>0}$ and let us prove that $f(z)$ is p-regular. We deduce from (8.13) that

$$f(z) = \sum_{i=0}^{\infty} \frac{(-1)^i}{\lambda^{i+1}} g(z^{p^i}).$$

The sum is well-defined because $\operatorname{supp} g \subset \mathbb{Z}[p^{-1}]_{>0}$. Let $N > 0$ be an integer such that $\operatorname{supp} f \subset \,]0, N]$ and let r be such that $\operatorname{supp} g(z^{p^r}) \subset]N, +\infty[$. Then, setting $g_i(z) = g(z^{p^i})$, we have

$$f(z) = \sum_{i=0}^{r-1} \frac{(-1)^i}{\lambda^{i+1}} (g_i)_{\leqslant N}(z).$$

Proposition 8.5 ensures that $g(z)$ is p-regular. Then it follows from Proposition 8.4 and Lemma 8.8 that the series $(g_i)_{\leqslant N}(z)$ are p-regular. Then, it follows from Proposition 8.7 that $f(z)$ is p-regular.

Let us now assume that the supports of $f(z)$ and $g(z)$ are included in $\mathbb{Z}[p^{-1}]_{<0}$. As above, up to replacing $f(z)$ and $g(z)$ by $f(z^{1/p^t})$ and $g(z^{1/p^t})$

respectively for some $t \in \mathbb{Z}_{\geqslant 0}$ large enough, one can assume that the supports of $f(z)$ and of $g(z)$ are included in $]-1/p, 0[\cap \mathbb{Z}[p^{-1}]_{<0}$. Consider $h(z) = z^{1/p} f(z)$ and $\theta(z) = zg(z)$. Note that the support of $\theta(z)$ is included in $]0,1[\cap \mathbb{Z}[p^{-1}]_{>0}$ and that $\theta(z)$ is p-regular thanks to Proposition 8.5. Note also that the support of $h(z)$ is included in $]0,1/p[\cap \mathbb{Z}[p^{-1}]_{>0}$. Therefore, $h^{\mathrm{rev}}(z)$ and $\theta^{\mathrm{rev}}(z)$ belong to $\overline{\mathbb{Q}}[[z]]$ and Lemma 8.9 ensures that $\theta^{\mathrm{rev}}(z)$ is p-regular. The equation (8.13) satisfied by $f(z)$ ensures that

$$h(z^p) + \lambda z^{(p-1)/p} h(z) = \theta(z).$$

It follows that, for all $\gamma \in \mathbb{Q}$,

$$h_\gamma + \lambda h_{p\gamma - \frac{p-1}{p}} = \theta_{p\gamma}. \tag{38}$$

This implies that, for all $\gamma \in \mathbb{Q}$,

$$h^{\mathrm{rev}}_\gamma + \lambda h^{\mathrm{rev}}_{p^{-1}\gamma-(p-1)} = \theta^{\mathrm{rev}}_{p^{-1}\gamma}.$$

Indeed, this follows from immediately from (38) using the following properties :

- $\mathrm{supp}(h(z)) \subset]0,1/p[\cap \mathbb{Z}[p^{-1}]_{>0}$;
- $\mathrm{supp}(h^{\mathrm{rev}}(z)) \subset \mathrm{rev}(]0,1/p[\cap \mathbb{Z}[p^{-1}]_{>0}) = p\mathbb{Z}_{\geqslant 1}$;
- $p^{-1}\gamma - (p-1) \in p\mathbb{Z}_{\geqslant 1} \Leftrightarrow p\,\mathrm{rev}(\gamma) - \frac{p-1}{p} \in]0,1/p[\cap \mathbb{Z}[p^{-1}]_{>0}$; moreover, in this case, $p\,\mathrm{rev}(\gamma) - \frac{p-1}{p} = \mathrm{rev}(p^{-1}\gamma - (p-1))$;
- $\mathrm{rev}(p^{-1}\gamma) = p\,\mathrm{rev}(\gamma)$.

Thus, we obtain

$$h^{\mathrm{rev}}(z) + \lambda z^{p(p-1)} h^{\mathrm{rev}}(z^p) = \theta^{\mathrm{rev}}(z^p)\,.$$

It follows from Lemma 8.14 below that $h^{\mathrm{rev}}(z)$ is p-regular. Lemma 8.9 guarantees that $h(z)$ is p-regular. So, $f(z)$ is quasi-p-regular. □

In the proof of the previous lemma, we have used the following result.

Lemma 8.14 ([Dum93, Theorem 24])**.** *Consider $f(z), g(z) \in \overline{\mathbb{Q}}[[z]]$ and $a(z) \in \overline{\mathbb{Q}}[z]$ such that*

$$f(z) + a(z) f(z^p) = g(z)\,.$$

If $g(z)$ is p-regular, then $f(z)$ is p-regular as well.

Proof of Proposition 8.11. We proceed by well-founded induction : consider an arbitrary $\boldsymbol{\omega} = (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in \boldsymbol{\Lambda}$ and suppose that, for any $\boldsymbol{\omega}' = (\boldsymbol{\alpha}', \boldsymbol{\lambda}', \boldsymbol{a}') \in \boldsymbol{\Lambda}$ such that $\boldsymbol{\alpha}' \prec \boldsymbol{\alpha}$, the Hahn series $\xi_{\boldsymbol{\omega}'}$ is quasi-p-regular. If $\boldsymbol{\omega} = ((), (), ())$, then $\xi_{\boldsymbol{\omega}} = 1$ is quasi-p-regular. Assume $\boldsymbol{\omega} \neq ((), (), ())$. Then, by Lemma 8.12 and by induction hypothesis, there exists $\lambda \in \overline{\mathbb{Q}}^\times$ such that $\xi_{\boldsymbol{\omega}}(z^p) - \lambda \xi_{\boldsymbol{\omega}}(z)$ is a linear combination of some quasi-p-regular Hahn series. Since the support of $\xi_{\boldsymbol{\omega}}$ is bounded, it follows from Lemma 8.13 that $\xi_{\boldsymbol{\omega}}$ is quasi-p-regular. This concludes the induction. □

8.4. **Quasi-p-regular series are Mahler.** In this subsection, we prove the following result.

Proposition 8.15. *Any quasi-p-regular Hahn series is p-Mahler.*

Proposition 8.15 is proved after two lemmas. In what follows, we let $\mathscr{F}$ be the set of formal series

$$f(z) = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma$$

with coefficients $f_\gamma \in \overline{\mathbb{Q}}$ and with support in $\mathbb{Z}[p^{-1}]_{\geqslant -N}$ for some $N \in \mathbb{Z}_{\geqslant 0}$. We let $\mathscr{F}_{\geqslant 0}$ be the set of $f \in \mathscr{F}$ with support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. The set $\mathscr{F}$ contains the set of Hahn series with support in $\mathbb{Z}[p^{-1}]$. Moreover, it has a natural structure of $\mathscr{P}$-module (and, hence, of $\mathbb{K}_\infty$-module, where $\mathbb{K}_\infty = \overline{\mathbb{Q}}(z^{\frac{1}{\star}})$) because the usual Cauchy product fg is well-defined for $f \in \mathscr{P}$ and $g \in \mathscr{F}$ (be careful, this Cauchy product is not well-defined for any pair of elements of $\mathscr{F}$). Furthermore, the map $\phi_p : \mathscr{H} \rightarrow \mathscr{H}$, which sends any $f \in \mathscr{H}$ to $f(z^p)$, has an obvious extension $\phi_p : \mathscr{F} \rightarrow \mathscr{F}$. It is thus meaningful to consider the p-Mahler elements of $\mathscr{F}$, *i.e.*, the solutions in $\mathscr{F}$ of p-Mahler equations.

We let $L_0 \subset \{0, \ldots, p-1\}^\star$ be the set consisting of the empty word together with all finite words $s_1 \cdots s_t$ over the alphabet $\{0, \ldots, p-1\}$ such that $t \geqslant 1$ and $s_1 \neq 0$. Note that we have $L_p = L_0 \blacksquare \operatorname{rev}(L_0)$. For any $w = s_1 \cdots s_t \in L_0$, we let

$$|w| = s_1 p^{t-1} + s_2 p^{t-2} + \cdots + s_t\,,$$

so that, for any $w_1 \blacksquare w_2 \in L_p$, $\|w_1 \blacksquare w_2\| = |w_1| + \|\blacksquare w_2\|$.

Lemma 8.16. *Let $f(z)$ be a p-regular Hahn series and let $(\mu, \boldsymbol{\tau}, \boldsymbol{\lambda})$ be an associated linear representation. We have*

$$f(z) = \boldsymbol{\tau} F_1(z) \mu(\blacksquare) F_2(z) \boldsymbol{\lambda}$$

with

$$F_1(z) = \sum_{w \in L_0} \mu(w) z^{|w|} \in \mathrm{M}_m(\overline{\mathbb{Q}}[[z]])$$

and

$$F_2(z) = \sum_{w \in \operatorname{rev}(L_0)} \mu(w) z^{\|\blacksquare w\|} \in \mathrm{M}_m(\mathscr{F}_{\geqslant 0}).$$

Proof. Write $f(z) = \sum_{\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}} f_\gamma z^\gamma$. We have

$$\begin{aligned}
\boldsymbol{\tau} F_1(z) \mu(\blacksquare) F_2(z) \boldsymbol{\lambda} &= \boldsymbol{\tau} \sum_{w_1 \in L_0} \mu(w_1) z^{|w_1|} \mu(\blacksquare) \sum_{w_2 \in \operatorname{rev}(L_0)} \mu(w_2) z^{\|\blacksquare w_2\|} \boldsymbol{\lambda} \\
&= \sum_{w_1 \in L_0, w_2 \in \operatorname{rev}(L_0)} \boldsymbol{\tau} \mu(w_1 \blacksquare w_2) \boldsymbol{\lambda} z^{|w_1| + \|\blacksquare w_2\|} \\
&= \sum_{w_1 \in L_0, w_2 \in \operatorname{rev}(L_0)} f_{\|w_1 \blacksquare w_2\|} z^{\|w_1 \blacksquare w_2\|} \\
&= \sum_{\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}} f_\gamma z^\gamma = f(z)\,.
\end{aligned}$$

□

Lemma 8.17. *Let* $\boldsymbol{f}(z) \in \mathscr{F}_{\geqslant 0}^{1\times m}$, $B(z) \in \overline{\mathbb{Q}}[z]^{m\times m}$ *and* $\boldsymbol{r}(z) \in \overline{\mathbb{Q}}[z]^{1\times m}$ *such that either*

$$\boldsymbol{f}(z) = \boldsymbol{f}(z^p)B(z) + \boldsymbol{r}(z) \text{ or } \boldsymbol{f}(z^p) = \boldsymbol{f}(z)B(z) + \boldsymbol{r}(z)\,.$$

Then, the entries of $\boldsymbol{f}(z)$ *are* p*-Mahler.*

Proof. Let us first assume that $\boldsymbol{f}(z^p) = \boldsymbol{f}(z)B(z) + \boldsymbol{r}(z)$. Then, the finite dimensional $\overline{\mathbb{Q}}(z)$-vector space V generated by the entries of $\boldsymbol{f}(z)$ and by 1 is invariant under ϕ_p. Therefore, for any $g \in V$, the family $(\phi_p^k(g))_{k\in\mathbb{Z}_{\geqslant 0}}$ is $\overline{\mathbb{Q}}(z)$-linearly dependent and, hence, g is p-Mahler. Since the entries of $\boldsymbol{f}$ belongs to V, this concludes the proof.

Let us now assume that $\boldsymbol{f}(z) = B(z)\boldsymbol{f}(z^p) + \boldsymbol{r}(z)$. We have $\boldsymbol{f}(z^{\frac{1}{p}}) = B(z^{\frac{1}{p}})\boldsymbol{f}(z) + \boldsymbol{r}(z^{\frac{1}{p}})$. Then, the finite dimensional $\mathbb{K}_\infty$-vector space W generated by the entries of $\boldsymbol{f}(z)$ and by 1 is invariant under ϕ_p^{-1}. Therefore, for any $g \in W$, the family $(\phi_p^{-k}(g))_{k\in\mathbb{Z}_{\geqslant 0}}$ is $\mathbb{K}_\infty$-linearly dependent and, hence, g is p-Mahler. Since the entries of $\boldsymbol{f}$ belongs to W, this concludes the proof. ☐

Proof of Proposition 8.15. Consider $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that $z^\nu f(z^d)$ is p-regular. Since $f(z)$ is p-Mahler if and only if $z^\nu f(z^d)$ is p-Mahler, we can assume that $f(z)$ is p-regular. Since "being p-Mahler" is a property closed under sums and products, in order to prove that f is p-Mahler, it is sufficient to prove that the entries of the matrices $F_1(z)$ and $F_2(z)$ given by Lemma 8.16 are p-Mahler. Let ϵ denote the empty word in L_0. The map $(w, s) \mapsto ws$ induces a bijection between $L_0 \times \{0, \ldots, p-1\}$ and $\{0\} \cup L_0 \backslash \{\epsilon\}$. Thus, we have

$$\begin{aligned} F_1(z) = \mu(\epsilon) + \sum_{w\in L_0\backslash\{\epsilon\}} \mu(w) z^{|w|} &= \mu(\epsilon) - \mu(0) + \sum_{s=0}^{p-1} \sum_{w\in L_0} \mu(w)\mu(s) z^{p|w|+s} \\ &= F_1(z^p)B(z) + \mu(\epsilon) - \mu(0) \end{aligned}$$

where $B(z) = \sum_{s=0}^{p-1} \mu(s) z^s$. It follows from Lemma 8.17 applied to each row of $F_1(z)$ that the entries of $F_1(z)$ are p-Mahler series.

Similarly, we have

$$\begin{aligned} F_2(z^p) &= \mu(\epsilon) + \sum_{w\in \mathrm{rev}(L_0)\backslash\{\epsilon\}} \mu(w) z^{p\|\blacksquare w\|} \\ &= \mu(\epsilon) - \mu(0) + \sum_{s=0}^{p-1} \sum_{w\in\mathrm{rev}(L_0)} \mu(s)\mu(w) z^{\|\blacksquare w\|+s} \\ &= C(z)F_2(z) + \mu(\epsilon) - \mu(0) \end{aligned}$$

where $C(z) = \sum_{s=0}^{p-1} \mu(s) z^s$. Thus,

$$F_2(z^p)^\top = F_2(z)^\top C(z)^\top + \mu(\epsilon)^\top - \mu(0)^\top$$

and it follows from Lemma 8.17 applied to each row of $F_2(z)^\top$ that the entries of $F_2(z)$ are p-Mahler. ☐

8.5. **Proof of Case (Reg) of Theorem 1.6.** We are almost ready to prove the first case in Theorem 1.6. We just need two more lemmas.

Lemma 8.18. *Fix* $N \in \mathbb{Z}_{\geqslant 0}$. *Consider the maps* $\alpha_N : \mathbb{Z}[p^{-1}]_{\geqslant 0} \to p^N\mathbb{Z}_{\geqslant 0}$ *and* $\beta_N : \mathbb{Z}[p^{-1}]_{\geqslant 0} \to \mathbb{Z}[p^{-1}] \cap [0, p^N[$ *uniquely defined by the equality*

$$\gamma = \alpha_N(\gamma) + \beta_N(\gamma).$$

We extend these maps to $\mathbb{Q}$, *setting* $\alpha_N(\gamma) = \beta_N(\gamma) = \gamma$ *when* $\gamma \in \mathbb{Q}\backslash\mathbb{Z}[p^{-1}]_{\geqslant 0}$. *If* $(a_\gamma)_{\gamma\in\mathbb{Q}} \in \overline{\mathbb{Q}}^{\mathbb{Q}}$ *is* p*-regular, then both* $(a_{\alpha_N(\gamma)})_{\gamma\in\mathbb{Q}}$ *and* $(a_{\beta_N(\gamma)})_{\gamma\in\mathbb{Q}}$ *are* p*-regular.*

Proof. We only prove the assertion concerning $(a_{\alpha_N(\gamma)})_{\gamma\in\mathbb{Q}}$, the proof for $(a_{\beta_N(\gamma)})_{\gamma\in\mathbb{Q}}$ being similar. Using the equality $\alpha_N(\gamma) = p^N\alpha_0(\gamma/p^N)$ and Lemma 7.13, we see that it is sufficient to prove the result for $N = 0$. From now on, we assume that $N = 0$. Recall that L_0 denotes the regular language consisting of the empty word together with all finite words $s_1 \cdots s_t$ over the alphabet $\{0, \ldots, p-1\}$ such that $t \geqslant 1$ and $s_1 \neq 0$. Then, the language $L = L_0\blacksquare$ on the alphabet $\{0, \ldots, p-1, \blacksquare\}$ is regular and it follows from Lemma 7.4 that the series $a_L = \sum_{w\in L_0} a_{\|w\blacksquare\|}w\blacksquare$ is rational. Moreover, the language $\mathrm{rev}(L_0)$ is regular (see [AS03, Corollary 4.3.5]) and, hence, its characteristic series, say b, is rational. Thus, the series $a_L b$ is rational. But, we have

$$a_L b = \sum_{v,w\in L_0} a_{\|v\blacksquare\|}(v \blacksquare \mathrm{rev}(w))$$

and, since $L_p = L\,\mathrm{rev}(L_0)$,

$$a_L b = \sum_{\gamma\in\mathbb{Z}[p^{-1}]_{\geqslant 0}} a_{\alpha_0(\gamma)}[\gamma]_p.$$

Thus, $(a_{\alpha_0(\gamma)})_{\gamma\in\mathbb{Q}}$ is p-regular. ☐

Lemma 8.19. *Let* $f \in \mathscr{P}$ *be a quasi-*p*-regular Puiseux series and* $g \in \mathscr{H}$ *be a quasi-*p*-regular Hahn series with bounded support. Then* fg *is quasi-*p*-regular.*

Proof. Let us first assume that $f \in \overline{\mathbb{Q}}[[z]]$ and that the support of g is a subset of $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. Note that Proposition 8.5 ensures that f and g are p-regular. Let us prove that fg is p-regular.

We first claim that, letting N be an arbitrary positive integer, it is sufficient to prove this for all $f \in \overline{\mathbb{Q}}[[z^{p^N}]]$ instead of all $f \in \overline{\mathbb{Q}}[[z]]$. To see this, consider the decomposition

$$f = \sum_{i=0}^{p^N-1} f_i(z^{p^N})z^i$$

with $f_i(z) \in \overline{\mathbb{Q}}[[z]]$. We have

$$fg = \sum_{i=0}^{p^N-1} f_i(z^{p^N})z^i g.$$

According to this formula and to Proposition 8.7, our claim will be established if we can prove that, for all $i \in \{0, \ldots, p^N - 1\}$, the series $f_i(z^{p^N}) \in$

$\overline{\mathbb{Q}}[[z^{p^N}]]$ is p-regular. To prove this, let us first note that $z^i f_i(z^{p^N})$ is the Hadamard product of $f(z)$ with the characteristic series $\chi_i(z)$ of the language given by

$$\mathcal{L}_i = \{w \in L_p \mid \|w\| = kp^N + i \text{ for some } k \in \mathbb{Z}_{\geqslant 0}\}.$$

But, $f(z)$ is p-regular and $\chi_i(z)$ is p-regular as well because $\mathcal{L}_i$ is a regular language. Now, the fact that $f_i(z^{p^N})$ is p-regular follows from the fact that p-regularity is invariant by Hadamard products in virtue of Proposition 8.7.

So, we now assume that $f \in \overline{\mathbb{Q}}[[z^{p^N}]]$. We choose $N \in \mathbb{Z}_{\geqslant 0}$ so that the support of g is included in $[0, p^N] \cap \mathbb{Z}[p^{-1}]_{\geqslant 0}$ (this is possible because the support of g is included in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$ and bounded by assumption). Setting $f(z) = \sum_{\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}} f_\gamma z^\gamma$ and $g(z) = \sum_{\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}} g_\gamma z^\gamma$, our choice of N guarantees that

$$f(z)g(z) = \sum_{\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}} f_{\alpha_N(\gamma)} g_{\beta_N(\gamma)} z^\gamma$$

where $\alpha_N(\gamma)$ and $\beta_N(\gamma)$ are defined in Lemma 8.18. But it follows from Lemma 8.18 that $(f_{\alpha_N(\gamma)})_{\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}}$ and $(g_{\beta_N(\gamma)})_{\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}}$ are p-regular and it follows from Proposition 7.11 that $(f_{\alpha_N(\gamma)} g_{\beta_N(\gamma)})_{\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}}$ is p-regular. So, fg is p-regular. This concludes the proof of the lemma under the assumption that $f \in \overline{\mathbb{Q}}[[z]]$ and that the support of g is a subset of $\mathbb{Z}[p^{-1}]_{\geqslant 0}$.

We now come to the general case. So, we let f be a quasi-p-regular Puiseux series and g be a quasi-p-regular Hahn series with bounded support. Consider $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that $z^\nu f(z^d) \in \overline{\mathbb{Q}}[[z]]$ and such that the support of $z^\nu g(z^d)$ is included in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. Since f and g are quasi-p-regular, Proposition 8.4 implies that $z^\nu f(z^d)$ and $z^\nu g(z^d)$ are p-regular. It follows from the first part of the proof that $z^{2\nu}(fg)(z^d) = (z^\nu f(z^d))(z^\nu g(z^d))$ is p-regular. Therefore, fg is quasi-p-regular. ☐

Proof of Case (Reg) *of Theorem* 1.6. Suppose that $f(z)$ is a p-Mahler Hahn series satisfying $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ or $(\mathcal{O}\Omega_5)$. In particular, $f(z)$ satisfies $(\mathcal{O}_3)$ and Proposition 6.2 implies that

$$f = \sum_{\boldsymbol{\omega} \in E} f_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}}\,,$$

where $E \subset \boldsymbol{\Lambda}_{\mathrm{st}}$ is a finite set and the $f_{\boldsymbol{\omega}}$ are p-Mahler Puiseux series satisfying $(\mathcal{O}_3)$. By Corollary 8.10, the $f_{\boldsymbol{\omega}}$ are quasi-p-regular. Furthermore, by Proposition 8.11, the $\xi_{\boldsymbol{\omega}}$ are quasi-p-regular Puiseux series. Since the supports of the $\xi_{\boldsymbol{\omega}}$ are bounded, we infer from Lemma 8.19 that the products $f_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}}$ are quasi-p-regular. Proposition 8.7 ensures that f is quasi-p-regular.

Conversely, suppose that $f(z) = \sum_\gamma f_\gamma z^\gamma$ is quasi-p-regular. Since, for any $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$, we have that $f(z)$ satisfies $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ or $(\mathcal{O}\Omega_5)$ if and only if $z^\nu f(z^d)$ satisfies the same property, we can assume that $f(z)$ is p-regular. Let $(\mu, \boldsymbol{\tau}, \boldsymbol{\lambda})$ be a linear representation associated to $f(z)$. On the one hand, for any $\gamma \in \mathbb{Q}$, we have $\mathrm{den}(\gamma) = p^u$, where $[\gamma]_p = s_1 \cdots s_t \blacksquare s_{t+1} \cdots s_u$. Thus, $\mathscr{h}(\gamma) \geqslant u$. On the other hand, it follows from (9) that

$$\mathscr{h}(\boldsymbol{\tau}\mu(s_1 \cdots s_t \blacksquare s_{t+1} \cdots s_u)\boldsymbol{\lambda}) = \mathcal{O}(u),$$

where the underlying constant depends only on the height of the entries of $\boldsymbol{\tau}$, $\boldsymbol{\lambda}$ and of the matrices $\mu(0), \ldots, \mu(p-1), \mu(\blacksquare)$. Then,

$$\hbar(f_\gamma) = \hbar(\boldsymbol{\tau}\mu(s_1 \cdots s_t \blacksquare s_{t+1} \cdots s_u)\boldsymbol{\lambda}) = \mathcal{O}(u) = \mathcal{O}(\hbar(\gamma))$$

and $f(z)$ satisfies ($\mathcal{O}_3$). Moreover, Proposition 8.15 ensures that f is p-Mahler. Then, it follows from Theorem 1.4 that $f(z)$ satisfies ($\mathcal{O}\Omega_3$), ($\mathcal{O}\Omega_4$) or ($\mathcal{O}\Omega_5$). ☐

Let us note the following consequence of Case (Reg) of Theorem 1.6.

Corollary 8.20. *The set of p-regular Hahn series is a $\overline{\mathbb{Q}}[z]$-algebra. The set of quasi-p-regular Hahn series is a $\mathbb{K}_\infty$-algebra.*

Proof. In light of Proposition 8.7, it only remains to prove that the product of two p-regular (*resp.* quasi-p-regular) series is p-regular (*resp.* quasi-p-regular). Let $f, g \in \mathscr{H}$ be quasi-p-regular series. It follows from Case (Reg) of Theorem 1.6 that they satisfy ($\mathcal{O}_3$). Consider the standard decomposition (7) of f and g, say

$$f = \sum_{\boldsymbol{\omega} \in E} f_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}}, \quad \text{and} \quad g = \sum_{\boldsymbol{\omega}' \in E'} g_{\boldsymbol{\omega}'} \xi_{\boldsymbol{\omega}'}\,,$$

where $E, E' \subset \boldsymbol{\Lambda}_{\mathrm{st}}$ are finite sets. Theorem 5.1 implies that the Puiseux series $f_{\boldsymbol{\omega}}$ and $g_{\boldsymbol{\omega}'}$ satisfy ($\mathcal{O}_3$). Therefore, these Puiseux series are quasi-p-regular. Since a product of p-regular power series is p-regular [AS92, Corollary 3.2], a product of quasi-p-regular Puiseux series is quasi-p-regular. Thus, each product $f_{\boldsymbol{\omega}} g_{\boldsymbol{\omega}'}$ is quasi-p-regular. We claim that each product $\xi_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}'}$ is quasi-p-regular. Indeed, it is easily seen that $\xi_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}'}$ is a linear combination over $\overline{\mathbb{Q}}[z^{-\frac{1}{\star}}]$ of some Hahn series $\xi_{\boldsymbol{\omega}''}$ with $\boldsymbol{\omega}'' \in \boldsymbol{\Lambda}$; see [FR25, Proposition 13]. Since the latter are quasi-p-regular by Proposition 8.11 and since the set of quasi-p-regular Hahn series is a $\overline{\mathbb{Q}}$-vector space, each product $\xi_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}'}$ is quasi-p-regular. Now, using Lemma 8.19, we get that the products

$$f_{\boldsymbol{\omega}} g_{\boldsymbol{\omega}'} \xi_{\boldsymbol{\omega}} \xi_{\boldsymbol{\omega}'}$$

with $(\boldsymbol{\omega}, \boldsymbol{\omega}') \in E \times E'$ are quasi-p-regular and, hence, fg is quasi-p-regular.

Suppose now that f and g are p-regular. Then, fg is quasi-p-regular by the first part of the proof. Furthermore, it has support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. It follows from Proposition 8.4 that fg is p-regular, as wanted. ☐

8.6. **Proof of Case (Aut) of Theorem 1.6.**

Recall that deterministic finite automata (DFAO) are defined in Section 7.4. We will freely use the terminology and notations introduced there.

Definition 8.21. *Consider a Hahn series $f(z) = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$.*

- *We say that $f(z)$ is p-automatic if* $\mathrm{supp}(f) \subset \mathbb{Z}[p^{-1}]_{\geqslant 0}$ *and if there exists a DFAO M with input alphabet $\Sigma_{\blacksquare,p}$ such that, for all $\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}$,*
$$f_\gamma = g_M([\gamma]_p).$$
- *We say that $f(z)$ is quasi-p-automatic if there exist $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that $z^\nu f(z^d)$ is p-automatic.*

Proposition 8.22. *We have the following properties relative to $f(z) \in \mathscr{H}$:*

- *$f(z)$ is p-automatic if and only if it is p-regular and its coefficients take finitely many values;*
- *$f(z)$ is quasi-p-automatic if and only if it is quasi-p-regular and its coefficients take finitely many values.*

Proof. The first statement follows immediately from Proposition 7.21.

The second statement follows from the fact that the following properties are equivalent:

1. $f(z)$ is quasi-p-automatic;
2. there exist $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that $z^{\nu} f(z^d)$ is p-automatic;
3. there exist $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ such that $z^{\nu} f(z^d)$ is p-regular and its coefficients take only finitely many values;
4. $f(z)$ is quasi-p-regular and the coefficients of $f(z)$ take only finitely many values;

all the equivalences being easy, except the equivalence between the second and third statements which follows from the first part of the proof. ☐

Remark 8.23. *If $f(z) \in \mathscr{H}$ is p-automatic, we may choose the DFAO given in Definition 8.21 so that $g_M(w) = 0$ for any $w \notin L_p$. Indeed, the regularity of the language L_p (Proposition 7.6) implies that there is a DFAO whose output function is the characteristic function of L_p. Then the result follows from* [AS03, Corollary 5.4.5] *or from Propositions 7.1 and 7.21.*

We are now ready to prove the last part of Theorem 1.6.

Proof of Case (Aut) of Theorem 1.6. Suppose that $f(z)$ is a p-Mahler Hahn series satisfying $(\mathcal{O}\Omega_5)$. It follows from Case (Reg) of Theorem 1.6 that $f(z)$ is quasi-p-regular. Since $f(z)$ satisfies $(\mathcal{O}\Omega_5)$, the coefficients of $f(z)$ take finitely many values. Thus, it follows from Proposition 8.22 that $f(z)$ is quasi-p-automatic.

Conversely, suppose that $f(z)$ is a quasi-p-automatic Hahn series. Then, Proposition 8.22 ensures that the coefficients of $f(z)$ take finitely many values and, hence, it satisfies $(\mathcal{O}\Omega_5)$. ☐

8.7. Minimal linear representation and the growth behaviors $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ and $(\mathcal{O}\Omega_5)$.

It is shown in [ABS23] that one can distinguish between properties $(\mathcal{O}\Omega_3)$, $(\mathcal{O}\Omega_4)$ and $(\mathcal{O}\Omega_5)$ for a given p-regular power series $f(z) \in \overline{\mathbb{Q}}[[z]]$ by looking at a minimal linear representation associated with $f(z)$. The aim of this section is to extend this result to Hahn series.

Let $f(z) \in \mathscr{H}$ be p-regular and let $(\mu, \boldsymbol{\tau}, \boldsymbol{\lambda})$ be a linear representation associated to $f(z)$. We say that this representation is minimal if there is no linear representation associated to $f(z)$ of smaller rank.

Let $m \geqslant 1$ be an integer and let Γ be a monoid of $m \times m$ square matrices with entries in $\overline{\mathbb{Q}}$. Following [ABS23], we say that Γ is *tame* if every eigenvalue of every element of Γ is either 0 or a root of unity. Note that any finite monoid of matrices is tame.

Theorem 8.24. *Let $f(z) \in \mathscr{H}$ be a quasi-p-regular Hahn series and let $\nu \in \mathbb{Z}_{\geqslant 0}$ and $d \in \mathbb{Z}_{\geqslant 1}$ be such that $z^{\nu} f(z^d)$ has support in $\mathbb{Z}[p^{-1}]_{\geqslant 0}$. Let $(\mu, \boldsymbol{\tau}, \boldsymbol{\lambda})$ be a minimal linear representation associated with $z^{\nu} f(z^d)$. Then, the following hold:*

- *$f(z)$ satisfies $(\mathcal{O}\Omega_3)$ if and only if the monoid $\mu(\Sigma^{\star}_{\blacksquare,p})$ is not tame;*
- *$f(z)$ satisfies $(\mathcal{O}\Omega_4)$ if and only if the monoid $\mu(\Sigma^{\star}_{\blacksquare,p})$ is tame but not finite;*
- *$f(z)$ satisfies $(\mathcal{O}\Omega_5)$ if and only if the monoid $\mu(\Sigma^{\star}_{\blacksquare,p})$ is finite.*

Proof. Set $g(z) = z^{\nu} f(z^d) = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma$. By Definitions 7.10 and 8.1, we have

$$g_\gamma = \boldsymbol{\tau}\mu([\gamma]_p)\boldsymbol{\lambda}$$

for all $\gamma \in \mathbb{Z}[p^{-1}]_{\geqslant 0}$ and $\boldsymbol{\tau}\mu(w)\boldsymbol{\lambda} = 0$ for any $w \in \Sigma^{\star}_{\blacksquare,p}\backslash L_p$.

Suppose that $\mu(\Sigma^{\star}_{\blacksquare,p})$ is not tame. Our proof is an adaptation of that of [ABS23, Lemma 8.4]. Since $\mu(\Sigma^{\star}_{\blacksquare,p})$ is not tame, there exists a word $w_0 \in \Sigma^{\star}_{\blacksquare,p}$ such that $\mu(w_0)$ has an eigenvalue ρ which is not a root of unity. Let $\boldsymbol{v} \in \overline{\mathbb{Q}}^m$ be such that $\mu(w_0)\boldsymbol{v} = \rho\boldsymbol{v}$. Since the representation is minimal, there exist words $w_1, \dots, w_m \in \Sigma^{\star}_{\blacksquare,p}$ such that $\mu(w_1)\boldsymbol{\lambda}, \dots, \mu(w_m)\boldsymbol{\lambda}$ form a basis of $\overline{\mathbb{Q}}^m$. Let $\kappa_1, \dots, \kappa_m \in \overline{\mathbb{Q}}$ be such that $\boldsymbol{v} = \kappa_1\mu(w_1)\boldsymbol{\lambda} + \cdots + \kappa_m\mu(w_m)\boldsymbol{\lambda}$. Again, by minimality, there exists a word $w \in \Sigma^{\star}_{\blacksquare,p}$ such that $\boldsymbol{\tau}\mu(w)\boldsymbol{v} \neq 0$. Then, for any $k \in \mathbb{Z}_{\geqslant 1}$,

$$\sum_{i=1}^{m} \kappa_i \boldsymbol{\tau}\mu(w w_0^k w_i)\boldsymbol{\lambda} = \boldsymbol{\tau}\mu(w)\mu(w_0)^k\boldsymbol{v} = \rho^k\boldsymbol{\tau}\mu(w)\boldsymbol{v} \neq 0.$$

Since ρ is not a root of unity, there exists a place v such that $|\rho|_{\mathsf{v}} > 1$. Then, there exists a positive real number c_0 and an integer $i \in \{1, \dots, m\}$ such that

$$|\boldsymbol{\tau}\mu(w w_0^k w_i)\boldsymbol{\lambda}|_{\mathsf{v}} \geqslant c_0|\rho|_{\mathsf{v}}^k \tag{39}$$

for any k in an infinite set $\mathcal{E} \subset \mathbb{Z}_{\geqslant 1}$. For any $k \in \mathcal{E}$, we have $\boldsymbol{\tau}\mu(w w_0^k w_i)\boldsymbol{\lambda} \neq 0$ and, hence, $w w_0^k w_i \in L_p$. For any $k \in \mathcal{E}$, we set $\gamma_k = \|w w_0^k w_i\|$. It follows from (39) that there exists $c_1 > 0$ such that, for any $k \in \mathcal{E}$, $\hslash(g_{\gamma_k}) \geqslant c_1 k$. Moreover, we have $\hslash(\gamma_k) = \mathcal{O}(k)$. Therefore, there exists $c_2 > 0$ such that, for all $k \in \mathcal{E}$,

$$\hslash(g_{\gamma_k}) \geqslant c_2\hslash(\gamma_k).$$

Therefore, $g_\gamma \in \Omega(\hslash(\gamma))$. Moreover, since g is p-regular, we infer from Theorem 1.6 that it satisfies $(\mathcal{O}_3)$. Finally, g satisfies $(\mathcal{O}\Omega_3)$ and it follows from Lemma 3.2 that f satisfies $(\mathcal{O}\Omega_3)$.

Suppose now that $\mu(\Sigma^{\star}_{\blacksquare,p})$ is tame but not finite. Arguing as in the proof of [ABS23, Lemma 9.2], we obtain that the coefficients of g take infinitely many distinct values. Thus, g and, hence, f do not satisfy $(\mathcal{O}\Omega_5)$. Moreover, a straightforward adaptation of the proof of (b)⇒(c) in [ABS23, Theorem 8.3] shows that f satifies $(\mathcal{O}_4)$. Thus, we infer from Theorem 1.4 that f satisfies $(\mathcal{O}\Omega_4)$.

Finally, suppose that $\mu(\Sigma^{\star}_{\blacksquare,p})$ is finite. Then the set $\{g_\gamma \mid \gamma \in \mathbb{Q}\} = \{f_\gamma \mid \gamma \in \mathbb{Q}\}$ is finite and, hence, $f(z)$ satisfies $(\mathcal{O}\Omega_5)$. ☐

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