HAHN SERIES AND MAHLER EQUATIONS: ALGORITHMIC ASPECTS

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Abstract. Many articles have recently been devoted to Mahler equations, partly because of their links with other branches of mathematics such as automata theory. Hahn series (a generalization of the Puiseux series allowing arbitrary exponents of the indeterminate as long as the set that supports them is well-ordered) play a central role in the theory of Mahler equations. In this paper, we address the following fundamental question: is there an algorithm to calculate the Hahn series solutions of a given linear Mahler equation? What makes this question interesting is the fact that the Hahn series appearing in this context can have complicated supports with infinitely many accumulation points. Our (positive) answer to the above question involves among other things the construction of a computable well-ordered receptacle for the supports of the potential Hahn series solutions.

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1. Introduction

Let $\mathbf{K}$ be a field (of any characteristic and not necessarily algebraically closed). A linear Mahler equation with coefficients in $\mathbf{K}(z)$ is a functional equation of the form

$$a_n(z)y(z^{\ell^n}) + a_{n-1}(z)y(z^{\ell^{n-1}}) + \cdots + a_0(z)y(z) = 0 \tag{1}$$

for some $\ell \in \mathbb{Z}_{\geq 2}$, $n \in \mathbb{Z}_{\geq 0}$ and $a_0(z), \ldots, a_n(z) \in \mathbf{K}(z)$ with $a_0(z)a_n(z) \neq 0$.

These equations are named after K. Mahler who wrote influential papers on the arithmetic nature of the values taken by solutions of such equations at algebraic points; see [Mah29, Mah30a, Mah30b]. Since then, the theory

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has undergone many developments, in various directions and is nowadays a very active field of research with many facets. The interactions between the theory of Mahler equations and other fields of mathematics have been fruitful in recent years. This is well illustrated by the work of Shäfke and Singer in [SS19] which gives a new proof of a conjecture of Loxton and van der Poorten – previously established by Adamczewski and Bell in [AB17] – and, therefore, a new proof of Cobham's theorem in automata theory by using tools coming from the theory of functional equations. Here are some references [Kub77, LvdP78, Mas82, Ran92, Dum93, Bec94, Nis96, DF96, Zan98, CZ02, AS03, Pel09, Ngu11, Ngu12, Phi15, BCZ16, AB17, AF17, AF18, DHR18, CDDM18, BCCD19, Fer18, Ada19, SS19, Roq21, Pou21, ADH21, Roq22, FP22, ABS22].

The Hahn series play a fundamental role in the theory of Mahler equations. Let us first look at the following simple but instructive example:

$$z^\ell f(z^{\ell^2}) - (z^\ell + z) f(z^\ell) + z f(z) = 0. \tag{2}$$

This equation has the obvious constant solution $f_1(z) = 1$ and any other solution in the field of formal Laurent series $\mathbf{K}((z))$ or even in the field of Puiseux series $\mathscr{P} = \bigcup_{d \in \mathbb{Z}_{\geq 1}} \mathbf{K}((z^{\frac{1}{d}}))$ is of the form $\lambda f_1(z)$ for some $\lambda \in \mathbf{K}$. However, a new solution can be found in the field $\mathscr{H}$ of Hahn series[1] with coefficients in $\mathbf{K}$ and value group $\mathbb{Q}$, namely

$$f_2(z) = \sum_{k \geq 1} z^{-\frac{1}{\ell^k}}. \tag{3}$$

Hence, working in the field of Hahn series, we have found two $\mathbf{K}$-linearly independent solutions of the above linear Mahler equation of order $n = 2$ (*i.e.*, as many $\mathbf{K}$-linearly independent solutions as the order of the equation), which is satisfactory.

Actually[2], when $\mathbf{K} = \overline{\mathbb{Q}}$, it follows from [Roq21] that the difference field $(\mathscr{H}, \phi_\ell)$, where ϕ_ℓ is the field automorphism of $\mathscr{H}$ sending $f(z)$ on $f(z^\ell)$, has a difference ring extension $(\mathcal{A}, \phi_\ell)$ such that

- for any $c \in \overline{\mathbb{Q}}^\times$, there exists $e_c \in \mathcal{A}$ satisfying $\phi_\ell(e_c) = c e_c$;
- there exists $l \in \mathcal{A}$ satisfying $\phi_\ell(l) = l + 1$;
- any linear Mahler equation of the form (1) has n $\overline{\mathbb{Q}}$-linearly independent solutions $y_1, \ldots, y_n \in \mathcal{A}$ of the form

$$y_i = \sum_{(c,j) \in \overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geq 0}} f_{i,c,j} e_c l^j \tag{4}$$

  where the sum is finite and the $f_{i,c,j}$ belong to $\mathscr{H}$.

This leads to the following fundamental question to which this article is devoted.

Question 1. *Is there an algorithm to calculate the Hahn series solutions of an equation of the form* (1)*?*

[1] See section 2 for the concept of Hahn series.

[2] In fact, the main results of [Roq21] and their proofs extend *mutatis mutandis* to an arbitrary algebraically closed field $\mathbf{K}$.

Before formulating this question more formally, let us say a few words about the calculation of the solutions of linear Mahler equations such as (1) in the more usual ring of formal power series $\mathbf{K}[[z]]$. By "calculating" the solutions of an equation of the form (1) in $\mathbf{K}[[z]]$, we usually mean calculating the formal power series solutions truncated to a specified order, *i.e.*, $N \in \mathbb{Z}_{\geq 0}$ being given, we want to determine the $\sum_{k\in\{0,\ldots,N\}} f_k z^k \in \mathbf{K}[z]$ for which there exists a solution

$$\widetilde{f}(z) = \sum_{k\in\mathbb{Z}_{\geq 0}} \widetilde{f}_k z^k \in \mathbf{K}[[z]]$$

of (1) such that

$$\sum_{k\in\{0,\ldots,N\}} \widetilde{f}_k z^k = \sum_{k\in\{0,\ldots,N\}} f_k z^k. \tag{5}$$

A natural formalization of Question 1 is obtained by replacing the sets of indices $\mathbb{Z}_{\geq 0}$ and $\{0,\ldots,N\}$ by $\mathbb{Q}$ and by an arbitrary finite subset $\mathcal{E}$ of $\mathbb{Q}$ respectively. More explicitly, this leads to the following formalization of Question 1: a finite subset $\mathcal{E}$ of $\mathbb{Q}$ being given, we want to determine the $\sum_{\gamma\in\mathcal{E}} f_\gamma z^\gamma \in \mathscr{H}$ for which there exists a solution

$$\widetilde{f}(z) = \sum_{\gamma\in\mathbb{Q}} \widetilde{f}_\gamma z^\gamma \in \mathscr{H}$$

of (1) such that

$$\sum_{\gamma\in\mathcal{E}} \widetilde{f}_\gamma z^\gamma = \sum_{\gamma\in\mathcal{E}} f_\gamma z^\gamma. \tag{6}$$

Remark 2. *(1) An arbitrary Hahn series truncated at an given order has infinitely many nonzero coefficients in general. For instance, the truncation at order* 0 *of the Hahn series* $f_2(z)$ *given by* (3) *is* $f_2(z)$ *itself and has infinitely many nonzero coefficients. This is why Question 1 is not stated in terms of truncated Hahn series.*

(2) The truncation $\sum_{k\in\{0,\ldots,N\}} \widetilde{f}_k z^k$ *of* $\widetilde{f}(z) = \sum_{k\in\mathbb{Z}_{\geq 0}} \widetilde{f}_k z^k \in \mathbf{K}[[z]]$ *can be interpreted as what remains of* $\widetilde{f}(z)$ *when only the indices* $k \in \mathbb{Z}_{\geq 0}$ *such that* $H(k) \leq N$ *are retained, where* H *denotes the naive height function defined, for any rational number* $x = a/b$ *where* $a \in \mathbb{Z}$, $b \in \mathbb{Z}\setminus\{0\}$ *are coprime, by* $H(x) = \max\{|a|,|b|\}$. *This leads to an alternative formulation of Question 1 similar to that given above but with* $\mathcal{E}$ *replaced by* $\mathcal{E}_N = \{\gamma \in \mathbb{Q} \mid H(\gamma) \leq N\}$. *Since* $\mathcal{E}_N$ *is a finite subset of* $\mathbb{Q}$ *and since any finite subset* $\mathcal{E}$ *of* $\mathbb{Q}$ *is a subset of* $\mathcal{E}_N$ *for some* $N \in \mathbb{Z}_{\geq 0}$, *the formulation of Question 1 given above is equivalent to this one.*

Note that, since we are not imposing any conditions on $\mathcal{E}$, the solution $\widetilde{f}(z)$ may not be uniquely determined by (6). Fortunately, there is a simple condition guaranteeing that $\widetilde{f}(z)$ is uniquely determined by (6): for this to be true, it suffices that $-\mathcal{S} \subset \mathcal{E}$ where $\mathcal{S}$ is the (finite and explicit) set of slopes of (1) defined in section 3; this follows directly from Corollary 13. So, $-\mathcal{S}$ can serve as a set of indices for "initial coefficients" of all Hahn series solutions of (1) and, in the special case $\mathcal{E} = -\mathcal{S}$, Question 1 aims to describe the possible "initial coefficients". However, we draw the reader's attention to the fact that, even if one knows explicitly the "initial part" $\sum_{\gamma\in -\mathcal{S}} \widetilde{f}_\gamma z^\gamma$ of a solution $\widetilde{f}(z) = \sum_{\gamma\in\mathbb{Q}} \widetilde{f}_\gamma z^\gamma \in \mathscr{H}$ of (1), it is not obvious at all to compute the value of $\widetilde{f}_\gamma$ for a given $\gamma \in \mathbb{Q} \setminus -\mathcal{S}$ from it, whence the importance of allowing an arbitrary finite set $\mathcal{E}$ in Question 1.

An algorithm to find the solutions in the field of Puiseux series $\mathscr{P} = \bigcup_{d\in\mathbb{Z}_{\geq 1}} \mathbf{K}((z^{\frac{1}{d}}))$ of a given Mahler equation has been given in [CDDM18][3]. It consists in bounding the ramification of these solutions in order to reduce the problem to the search of the solutions in a specific field of ramified Laurent series $\mathbf{K}((z^{\frac{1}{d}}))$ for an explicit $d \in \mathbb{Z}_{\geq 1}$. What makes the search of the Hahn series solutions interesting is precisely the fact that one cannot reduce the problem to the search of (ramified) Laurent series solutions: one has to deal with Hahn series that might have rather involved supports. The support $\operatorname{supp} f_2(z) = \{-\frac{1}{\ell^k} \mid k \in \mathbb{Z}_{\geq 1}\}$ of the Hahn series $f_2(z)$ given by (3) is one of the simplest support one can expect for a (non Puiseux) Hahn series solution of a linear Mahler equation. Much more complicated supports may arise. For instance, the Hahn series $f_2(z)^2$ satisfies a linear Mahler equation of order 3 and, if the characteristic of $\mathbf{K}$ is not equal to 2, its support $\operatorname{supp} f_2(z)^2 = \{-\frac{1}{\ell^k} - \frac{1}{\ell^{k'}} \mid k, k' \in \mathbb{Z}_{\geq 1}\}$ has infinitely many accumulation points, namely any element of $\{0\} \cup \{-\frac{1}{\ell^k} \mid k \in \mathbb{Z}_{\geq 1}\}$. These complicated supports induce many difficulties.

1.1. Outline of our answer to Question 1.

Our approach to answer Question 1 relies on the following two ingredients.

(1) We introduce a subset $\mathcal{V}$ of $\mathbb{Q}$ satisfying the following properties:
   - $\mathcal{V}$ contains the support of any Hahn series solution of (1);
   - $\mathcal{V}$ is well-ordered;
   - $\mathcal{V}$ is computable in the sense that there exists an algorithm to determine whether a given rational number belongs to $\mathcal{V}$ or not;
   - $\mathcal{V}$ satisfies a technical but important condition that we do not state here.

(2) A finite set $\mathcal{E} \subset \mathcal{V}$ being given, we show that we can compute algorithmically a finite subset $\mathcal{R}$ of $\mathcal{V}$ containing $\mathcal{E}$ such that, for any $f(z) = \sum_{\gamma\in\mathcal{R}} f_\gamma z^\gamma \in \mathscr{H}$, the following properties are equivalent:
   - there exists a solution $\widetilde{f}(z) = \sum_{\gamma\in\mathbb{Q}} \widetilde{f}_\gamma z^\gamma \in \mathscr{H}$ of (1) such that
   $$\sum_{\gamma\in\mathcal{R}} \widetilde{f}_\gamma z^\gamma = \sum_{\gamma\in\mathcal{R}} f_\gamma z^\gamma;$$
   - the support of the Hahn series
   $$a_n(z)f(z^{\ell^n}) + a_{n-1}(z)f(z^{\ell^{n-1}}) + \cdots + a_0(z)f(z) \tag{7}$$
   is disjoint from $\psi(\mathcal{R})$ where $\psi : \mathbb{Q} \to \mathbb{Q}$ is an explicit map defined in section 3.4.

This reduces Question 1 to a question of linear algebra. Indeed, up to multiplying (1) by a suitable nonzero polynomial, one can assume that the

[3]We mention for the interested reader that, when $\mathbf{K} \subset \mathbb{C}$, any Puiseux series solution is actually convergent; see [BR13, Lem. 4] for example.

$a_i(z)$ are polynomials. Then, one can compute an explicit family of linear maps $F_\delta : \mathbf{K}^{\mathcal{R}} \to \mathbf{K}$ such that, for any $f(z) = \sum_{\gamma\in\mathcal{R}} f_\gamma z^\gamma \in \mathscr{H}$,

$$a_n(z)f(z^{\ell^n}) + a_{n-1}(z)f(z^{\ell^{n-1}}) + \cdots + a_0(z)f(z) = \sum_{\delta\in\mathbb{Q}} F_\delta((f_\gamma)_{\gamma\in\mathcal{R}})z^\delta.$$

The fact that the support of (7) is disjoint from $\psi(\mathcal{R})$ is equivalent to the fact that, for all $\delta \in \psi(\mathcal{R})$, $F_\delta((f_\gamma)_{\gamma\in\mathcal{R}}) = 0$. This is an (explicit) system of linear equations in the $(f_\gamma)_{\gamma\in\mathcal{R}}$ that can be solved algorithmically. This solves Question 1.

1.2. **Organization of the paper.** In section 2, we recall basic definitions and properties of the Hahn series. In section 3, we first recall the notions of Newton polygons and of slopes. We then state and prove several results used elsewhere in the paper. In section 4, we give an algorithm to compute a set $\mathcal{V}$ having the properties listed in section 1.1 above. In sections 6 and 7, we give an algorithm to compute a set $\mathcal{R}$ having the properties listed in section 1.1 above. In section 8, we describe an algorithm that answers Question 1 in the affirmative. In section 9, we apply our main algorithm to a classical equation.

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2. The ring of Hahn series

We denote by $\mathscr{H}$ the field of Hahn series with coefficients in the field $\mathbf{K}$ and with value group $\mathbb{Q}$ (see [Hah07]). An element of $\mathscr{H}$ is an $(f_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbf{K}^{\mathbb{Q}}$ whose support

$$\operatorname{supp}(f_\gamma)_{\gamma\in\mathbb{Q}} = \{\gamma \in \mathbb{Q} \mid f_\gamma \neq 0\}$$

is well-ordered, *i.e.*, such that any nonempty subset of this support has a least element. An element $(f_\gamma)_{\gamma\in\mathbb{Q}}$ of $\mathscr{H}$ is usually (and will be) denoted by

$$f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma.$$

The sum and product of two elements $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma$ and $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma$ of $\mathscr{H}$ are respectively defined by

$$f + g = \sum_{\gamma\in\mathbb{Q}} (f_\gamma + g_\gamma) z^\gamma$$

and

$$fg = \sum_{\gamma\in\mathbb{Q}} \left(\sum_{\gamma'+\gamma''=\gamma} f_{\gamma'} g_{\gamma''} \right) z^\gamma.$$

(Note that there are only finitely many $(\gamma', \gamma'') \in \mathbb{Q}\times\mathbb{Q}$ such that $\gamma'+\gamma'' = \gamma$ and $f_{\gamma'}g_{\gamma''} \neq 0$.) For a proof that $\mathscr{H}$ endowed with this ring structure is a field, we refer to [Neu49, Th. 5.7].

Since the support of any Hahn series is well-ordered, one can define the z-adic valuation

$$\begin{array}{rcl} \mathrm{val}: \mathscr{H} & \to & \mathbb{Q}\cup\{+\infty\} \\ f & \mapsto & \mathrm{val}\, f = \min \mathrm{supp}\, f \end{array}$$

with the convention $\min\emptyset = +\infty$. It satisfies the usual properties of a valuation, namely :

- $\forall f \in \mathscr{H}$, $(\mathrm{val}\, f = +\infty \iff f = 0)$;
- $\forall f, g \in \mathscr{H}$,

$$\mathrm{val}(fg) = \mathrm{val}\, f + \mathrm{val}\, g \tag{8}$$

  and

$$\mathrm{val}(f+g) \geqslant \min\{\mathrm{val}\, f, \mathrm{val}\, g\}. \tag{9}$$

For any subset $\mathcal{Q}$ of $\mathbb{Q}$, we let $\mathscr{H}_{|\mathcal{Q}}$ be the $\mathbf{K}$-vector space of Hahn series with support in $\mathcal{Q}$, *i.e.*,

$$\mathscr{H}_{|\mathcal{Q}} = \{f \in \mathscr{H} \mid \mathrm{supp}\, f \subset \mathcal{Q}\}.$$

We have a natural $\mathbf{K}$-linear map

$$\begin{array}{rccc} \bullet_{|\mathcal{Q}}: & \mathscr{H} & \to & \mathscr{H}_{|\mathcal{Q}} \\ & f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma & \mapsto & f_{|\mathcal{Q}} := \sum_{\gamma\in\mathcal{Q}} f_\gamma z^\gamma. \end{array}$$

For any $f, g \in \mathscr{H}$ and any $\mathcal{Q} \subset \mathbb{Q}$, we will say that "$f = g$ on $\mathcal{Q}$" if $f_{|\mathcal{Q}} = g_{|\mathcal{Q}}$.

3. Newton polygons

Let ϕ_ℓ be the field automorphism of $\mathscr{H}$ sending $f(z)$ on $f(z^\ell)$. We denote by

$$\mathcal{D}_{\mathbf{K}[z]} = \mathbf{K}[z]\langle\phi_\ell\rangle$$

the Ore algebra of noncommutative polynomials with coefficients in $\mathbf{K}[z]$ such that, for all $f \in \mathbf{K}[z]$, $\phi_\ell f = \phi_\ell(f)\phi_\ell$. An element of $\mathcal{D}_{\mathbf{K}[z]}$ will be called a Mahler operator.

In what follows, we consider an inhomogeneous Mahler equation

$$a_n(z)y(z^{\ell^n}) + a_{n-1}(z)y(z^{\ell^{n-1}}) + \cdots + a_0(z)y = a_{-\infty}(z) \tag{10}$$

with $a_0(z), \ldots, a_n(z) \in \mathbf{K}[z]$ and $a_{-\infty}(z) \in \mathscr{H}$ such that $a_0(z)a_n(z) \neq 0$. This equation can be rewritten as

$$L(y) = a_{-\infty}$$

where

$$L = a_n\phi_\ell^n + a_{n-1}\phi_\ell^{n-1} + \cdots + a_0 \in \mathcal{D}_{\mathbf{K}[z]}. \tag{11}$$

3.1. Newton polygon.

Following [CDDM18], we define the Newton polygon $\mathcal{N}(L, a_{-\infty})$ of (10) as the lower convex hull of the set

$$\mathcal{P}(L, a_{-\infty}) = \{(\ell^i, j) \mid i \in \{-\infty, 0, \ldots, n\},\ j \in \mathrm{supp}\, a_i\} \subset \mathbb{R}^2$$

with the convention $\ell^{-\infty} = 0$. In other terms, $\mathcal{N}(L, a_{-\infty})$ is the convex hull of the set

$$\{(\ell^i, j) \mid i \in \{-\infty, 0, \ldots, n\},\ j \geq \mathrm{val}\, a_i\} \subset \mathbb{R}^2. \tag{12}$$

3.2. **Slopes.** The polygon $\mathcal{N}(L, a_{\infty})$ is delimited by two vertical half lines and by finitely many nonvertical vectors having pairwise distinct slopes, called the slopes of (10). The set of slopes of (10) will be denoted by $\mathcal{S}(L, a_{-\infty})$. The following result gives an useful characterization of these slopes.

Lemma 3. *The following properties relative to $\mu \in \mathbb{Q}$ are equivalent:*

(i) μ belongs to $\mathcal{S}(L, a_{-\infty})$;

(ii) there exist distinct $i_1, i_2 \in \{-\infty, 0, \ldots, n\}$ such that

$$\operatorname{val} a_{i_1} - \ell^{i_1}\mu = \operatorname{val} a_{i_2} - \ell^{i_2}\mu = \min_{i \in \{-\infty, 0, \ldots, n\}} \operatorname{val} a_i - \ell^i \mu. \tag{13}$$

Moreover, if μ belongs to $\mathcal{S}(L, a_{-\infty})$, then the equality (13) is satisfied if and only if $(\ell^{i_1}, \operatorname{val} a_{i_1})$ and $(\ell^{i_2}, \operatorname{val} a_{i_2})$ belong to the edge of slope μ of $\mathcal{N}(L, a_{-\infty})$.

Proof. The fact that μ satisfies (i) is equivalent to the fact that μ is the slope of a nonvertical edge of $\mathcal{N}(L, a_{-\infty})$ which is in turn equivalent to the fact that there exists $b \in \mathbb{R}$ such that the affine line $y - \mu x = b$ contains an edge of $\mathcal{N}(L, a_{-\infty})$.

But, since $\mathcal{N}(L, a_{-\infty})$ is the convex hull of the set (12), the affine line $y - \mu x = b$ contains an edge of $\mathcal{N}(L, a_{-\infty})$ if and only the following properties are satisfied:

- there exist distinct $i_1, i_2 \in \{-\infty, 0, \ldots, n\}$ such that $(\ell^{i_1}, \operatorname{val} a_{i_1})$ and $(\ell^{i_2}, \operatorname{val} a_{i_2})$ belong to the line $y - \mu x = b$;
- for any $i \in \{-\infty, 0, \ldots, n\}$, if $\operatorname{val} a_i < +\infty$, then $(\ell^i, \operatorname{val} a_i)$ belongs to the half space $y - \mu x \geq b$.

The latter two properties are of course equivalent to the fact that there exist distinct $i_1, i_2 \in \{-\infty, 0, \ldots, n\}$ such that

$$b = \operatorname{val} a_{i_1} - \ell^{i_1}\mu = \operatorname{val} a_{i_2} - \ell^{i_2}\mu = \min_{i \in \{-\infty, 0, \ldots, n\}} \operatorname{val} a_i - \ell^i \mu.$$

This shows that (i) and (ii) are equivalent.

Last, if μ belongs to $\mathcal{S}(L, a_{-\infty})$, then the previous discussion shows that the equation of the line containing the edge of slope μ of $\mathcal{N}(L, a_{-\infty})$ is $y - \mu x = b$ with $b = \min_{i \in \{-\infty, 0, \ldots, n\}} \operatorname{val} a_i - \ell^i \mu$. The last assertion of the lemma follows directly from this. □

3.3. **Newton polygon and slopes in the homogeneous case.** In the homogeneous case, that is when $a_{-\infty} = 0$, we will omit $a_{-\infty}$ in the previous notations and terminologies. For instance, $\mathcal{N}(L, 0)$ will simply be denoted by $\mathcal{N}(L)$ and will be called the Newton polygon of L.

Remark 4. *Note, for later use, the following simple but important fact: since the coefficients $a_0, \ldots, a_n$ of L are in $\mathbf{K}[z]$, the set $\mathcal{P}(L)$ is finite.*

We denote by

$$\mu_1 < \cdots < \mu_K$$

the slopes of L, so that

$$\mathcal{S}(L) = \{\mu_1, \ldots, \mu_K\}.$$

We let
$$p_0, \dots, p_K \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$$
be the vertices, ordered by increasing abscissa, of the polygon $\mathcal{N}(L)$. For any $k \in \{0, \dots, K\}$, we let α_k be the unique element of $\{0, \dots, n\}$ and β_k be the unique element of $\mathbb{Z}$ such that
$$p_k = (\ell^{\alpha_k}, \beta_k) = (\ell^{\alpha_k}, \operatorname{val} a_{\alpha_k}).$$
Note that $\alpha_0 = 0$ and that $\alpha_K = n$. With these notations, the edge of $\mathcal{N}(L)$ with slope μ_k has $p_{k-1} = (\ell^{\alpha_{k-1}}, \beta_{k-1})$ as its left endpoint and $p_k = (\ell^{\alpha_k}, \beta_k)$ as its right endpoint.

Example 5. The main algorithm presented in this paper will be illustrated in section 9 on the Mahler operator of order 2 given by
$$L = z\phi_2^2 + (z-1)\phi_2 - 2. \tag{14}$$
The set $\mathcal{P}(L)$, the Newton polygon $\mathcal{N}(L)$ and the vertices p_k associated to this specific L are given in section 9.1 and represented in Figure 2.

We note the following result for further use.

Lemma 6. *For all $(\ell^i, j) \in \mathcal{P}(L)$ and all $k \in \{1, \dots, K\}$, we have:*
$$-\ell^i \mu_k + j \geq -\ell^{\alpha_k}\mu_k + \beta_k = -\ell^{\alpha_{k-1}}\mu_k + \beta_{k-1}. \tag{15}$$
In geometric terms, this means that the minimum of the ordinates of the projections of points of $\mathcal{P}(L)$ along a line of slope μ_k on the y-axis is reached at p_{k-1} and at p_k.

Proof. Lemma 3 applied with $\mu = \mu_k$, $(\ell^{i_1}, \operatorname{val} a_{i_1}) = p_k$ and $(\ell^{i_2}, \operatorname{val} a_{i_2}) = p_{k-1}$ ensures that $-\ell^{\alpha_k}\mu_k + \beta_k = -\ell^{\alpha_{k-1}}\mu_k + \beta_{k-1} = \min_{i \in \{0,\dots,n\}} -\ell^i\mu_k + \operatorname{val} a_i$. The inequality (15) follows from this and from the fact that, for all $(\ell^i, j) \in \mathcal{P}(L)$, we have $j \geq \operatorname{val} a_i$ and, hence, $-\ell^i\mu_k + j \geq -\ell^i\mu_k + \operatorname{val} a_i$. □

3.4. **The maps Ψ, ψ and π.** In the rest of the paper, we will intensively use the following three maps:
$$\begin{array}{rcl} \Psi : \mathbb{Q} & \to & \{\text{Finite subsets of } \mathbb{Q}\} \\ v & \mapsto & \{v\ell^i + j \mid (\ell^i, j) \in \mathcal{P}(L)\} \end{array}$$
$$\begin{array}{rcl} \psi : \mathbb{Q} & \to & \mathbb{Q} \\ v & \mapsto & \min \Psi(v) = \min\{v\ell^i + j \mid (\ell^i, j) \in \mathcal{P}(L)\} \\ & & \quad = \min\{v\ell^i + \operatorname{val} a_i \mid i \in \{0, \dots, n\}\} \end{array} \tag{16}$$
and
$$\begin{array}{rcl} \pi : \mathbb{Q} & \to & \mathbb{Q} \\ q & \mapsto & \max\left\{\frac{q-j}{\ell^i} \mid (\ell^i, j) \in \mathcal{P}(L)\right\} = -\min \mathcal{S}(L, z^q)\,. \end{array}$$
These maps are well-defined because $\mathcal{P}(L)$ is finite according to Remark 4. In geometric terms:

- $\Psi(v)$ is the set of ordinates of the projection of the elements of $\mathcal{P}(L)$ along a line of slope $-v$ onto the y-axis;
- $\psi(v)$ is the least of these ordinates;
- $\pi(q)$ is the opposite of the minimum of the slopes of the lines passing through $(0, q)$ and an element of $\mathcal{P}(L)$.

Let us now give a more computational interpretation of these maps. Setting, for any $i \in \{0,\dots,n\}$,

$$a_i = \sum_{j\in \operatorname{supp} a_i} a_{i,j} z^j,$$

we have

$$L(z^v) = \sum_{i=0}^{n} a_{i,j} z^{v\ell^i + j} = \sum_{(\ell^i,j)\in \mathcal{P}(L)} a_{i,j} z^{v\ell^i+j}. \tag{17}$$

This formula shows that $\Psi(v)$ is a natural receptacle for the support of $L(z^v)$. Indeed, we have

$$\operatorname{supp} L(z^v) \subset \Psi(v), \tag{18}$$

this inclusion being an equality for all but finitely many $v \in \mathbb{Q}$, *e.g.*, for all $v \in \mathbb{Q}$ such that the exponents $v\ell^i + j$ involved in (17) are two by two distinct, because there is no cancellation between terms on the right-hand side of (17) in this case. It follows immediately from these remarks that

$$\psi(v) \leq \operatorname{val} L(z^v)$$

and that this inequality is an equality for all but finitely many $v \in \mathbb{Q}$ (actually, Lemma 11 below ensures that this is the case for all $v \in \mathbb{Q} \setminus -\mathcal{S}(L)$). Last, it is easily seen that, for all but finitely many $q \in \mathbb{Q}$, the equation $\operatorname{val} L(z^w) = q$ has a unique solution $w \in \mathbb{Q}$ and that it is given by $w = \pi(q)$.

It will be convenient to set

$$\psi(+\infty) = \pi(+\infty) = +\infty.$$

We shall now give several properties of the maps ψ and π which will shall use later.

Lemma 7. *The maps $\psi : \mathbb{Q} \to \mathbb{Q}$ and $\pi : \mathbb{Q} \to \mathbb{Q}$ are increasing*[4] *bijections and inverse of each other.*

Proof. The fact that these maps are increasing is immediate from their definitions. In particular, ψ and π are injective. In order to prove that ψ and π are inverse of each other, it is thus sufficient to prove that $\psi(\pi(q)) = q$ for every $q \in \mathbb{Q}$. Let us prove this. On the one hand, by definition of π, there exists $(\ell^i, j) \in \mathcal{P}(L)$ such that

$$\pi(q) = \frac{q-j}{\ell^i}.$$

Since $(\ell^i, j) \in \mathcal{P}(L)$, it follows from the definition of ψ that

$$\psi(\pi(q)) \leq \ell^i \pi(q) + j = \ell^i \frac{q-j}{\ell^i} + j = q.$$

On the other hand, by definition of ψ, there exists $(\ell^{i'}, j') \in \mathcal{P}(L)$ such that

$$\psi(\pi(q)) = \ell^{i'} \pi(q) + j'.$$

[4]In the whole paper, a function $f : \mathbb{Q} \to \mathbb{Q}$ is said to be increasing if, for all $x, y \in \mathbb{Q}$, $(y > x \Rightarrow f(y) > f(x))$. It is nondecreasing if, for all $x, y \in \mathbb{Q}$, $(y \geq x \Rightarrow f(y) \geq f(x))$. We will use similar terminologies for sequences of real numbers or of sets.

But, since $(\ell^{i'}, j') \in \mathcal{P}(L)$, it follows from the definition of π that

$$\pi(q) \geq \frac{q-j'}{\ell^{i'}}.$$

So, we have

$$\psi(\pi(q)) \geq \ell^{i'}\frac{q-j'}{\ell^{i'}} + j' = q.$$

Finally, we obtain $\psi(\pi(q)) = q$. □

Lemma 8. *Consider $v \in \mathbb{Q}$. If $k \in \{1, \ldots, K+1\}$ is such that $-\mu_k \leq v \leq -\mu_{k-1}$, with the conventions $\mu_0 = -\infty$ and $\mu_{K+1} = +\infty$, then*

$$\psi(v) = \ell^{\alpha_{k-1}} v + \beta_{k-1}\,. \tag{19}$$

In geometric terms, the formula (19) means that $\psi(v)$ is the ordinate of the projection of $p_{k-1} = (\ell^{\alpha_{k-1}}, \beta_{k-1})$ along a line of slope $-v$ on the y-axis.

Proof. By definition, $\psi(v) = \min\{\ell^\alpha v + \beta \mid (\ell^\alpha, \beta) \in \mathcal{P}(L)\}$ so, in order to prove the lemma, it is sufficient to prove that, for all $(\ell^\alpha, \beta) \in \mathcal{P}(L)$, for all $v \in [-\mu_k, -\mu_{k-1}]$, $\ell^\alpha v + \beta \geq \ell^{\alpha_{k-1}} v + \beta_{k-1}$. In other terms, we have to prove that, for $(\ell^\alpha, \beta) \in \mathcal{P}(L)$, the map

$$\begin{array}{rcl} \delta : \mathbb{R} & \rightarrow & \mathbb{R} \\ v & \mapsto & \ell^\alpha v + \beta - (\ell^{\alpha_{k-1}} v + \beta_{k-1}) \end{array}$$

takes nonnegative values on $[-\mu_k, -\mu_{k-1}]$. Let us prove this.

Let us first assume that $k \in \{2, \ldots, K\}$. It follows from Lemma 6 (applied with $k-1$ instead of k for the second inequality) that

$$-\ell^\alpha \mu_k + \beta \geq -\ell^{\alpha_{k-1}} \mu_k + \beta_{k-1} \quad \text{and} \quad -\ell^\alpha \mu_{k-1} + \beta \geq -\ell^{\alpha_{k-1}} \mu_{k-1} + \beta_{k-1}.$$

Thus $\delta(-\mu_k) \geq 0$ and $\delta(-\mu_{k-1}) \geq 0$. Since δ is affine, it follows that, for all $v \in [-\mu_k, -\mu_{k-1}]$, $\delta(v) \geq 0$, as wanted.

Let us now consider the case $k = 1$. It follows from Lemma 6 that

$$-\ell^\alpha \mu_1 + \beta \geq -\ell^{\alpha_0} \mu_1 + \beta_0.$$

In other words, $\delta(-\mu_1) \geq 0$. Since $\alpha_0 = 0$, the function δ is either increasing or constant. Thus, for any $v \geq -\mu_1$, $\delta(v) \geq \delta(-\mu_1) \geq 0$, as wanted.

Let us eventually consider the case $k = K+1$. It follows from Lemma 6 that

$$-\ell^\alpha \mu_K + \beta \geq -\ell^{\alpha_K} \mu_K + \beta_K.$$

In other words, $\delta(-\mu_K) \geq 0$. Since $\alpha_K = n \geq \alpha$, the function δ is either decreasing or constant. Thus, for any $v \leq -\mu_K$, $\delta(v) \geq \delta(-\mu_K) \geq 0$, as wanted. □

Example 9. An illustration of Lemma 8 for the operator L given by (14) is given by Figure 2 in Section 9. Indeed, with the hypotheses and notations of Figure 2, we have $K = 2$, $\mu_1 = 0$, $\mu_2 = 1/2$ and Lemma 8 ensures that:

- if $0 \leq v$, then $\psi(v)$ is the ordinate of the projection of p_0 along a line of slope $-v$ on the y-axis;
- if $-1/2 \leq v \leq 0$, then $\psi(v)$ is the ordinate of the projection of p_1 along a line of slope $-v$ on the y-axis;

- if $v \leq -1/2$, then $\psi(v)$ is the ordinate of the projection of p_2 along a line of slope $-v$ on the y-axis.

This is indeed what we see on Subfigure 2(A) (resp. 2(B), 2(C)) when $v = 1/4$ (resp. $-1/4$, $-3/4$).

Lemma 10. *If $-\mu_k \leq \pi(q) \leq -\mu_{k-1}$ for some $k \in \{1, \dots, K+1\}$ with the conventions $\mu_0 = -\infty$ and $\mu_{K+1} = +\infty$, then*

$$\pi(q) = \frac{q - \beta_{k-1}}{\ell^{\alpha_{k-1}}}. \tag{20}$$

In geometric terms, the formula (20) means that $\pi(q)$ is the opposite of the slope of the line passing through $(0, q)$ and $p_{k-1} = (\ell^{\alpha_{k-1}}, \beta_{k-1})$.

Proof. Applying Lemma 8 with $v = \pi(q)$ we obtain

$$\psi(\pi(q)) = \ell^{\alpha_{k-1}}\pi(q) + \beta_{k-1}.$$

The result follows from this formula since, according to Lemma 7, we have $q = \psi(\pi(q))$.

□

3.5. **Supports, valuations and the maps Ψ, ψ and π.** Roughly speaking, the results presented in this section aim to relate the valuations and the supports of $L(f)$ and of f *via* the maps Ψ, ψ and π introduced in section 3.4. These results will be used extensively in the remainder of this paper.

Lemma 11. *For any $f \in \mathscr{H}$, we have*

$$\operatorname{val} L(f) \geq \psi(\operatorname{val} f).$$

If $\operatorname{val} f \notin -\mathcal{S}(L)$, then

$$\operatorname{val} L(f) = \psi(\operatorname{val} f).$$

Proof. The result is obvious if $f = 0$. In the rest of the proof, we assume that $f \neq 0$. As a preliminary remark, note the following obvious but important formula:

$$\min_{i \in \{0, \dots, n\}} \operatorname{val}(a_i \phi_\ell^i(f)) = \min_{i \in \{0, \dots, n\}} \operatorname{val} a_i + \ell^i \operatorname{val} f = \psi(\operatorname{val} f). \tag{21}$$

We have

$$L(f) = a_n \phi_\ell^n(f) + a_{n-1}\phi_\ell^{n-1}(f) + \cdots + a_0 f. \tag{22}$$

Using (9), we get

$$\operatorname{val} L(f) \geq \min_{i \in \{0, \dots, n\}} \operatorname{val}(a_i \phi_\ell^i(f)).$$

Combining the latter inequality with (21), we get the first assertion of the Lemma.

Let us prove the contrapositive of the second assertion. Assume that $\operatorname{val} L(f) \neq \psi(\operatorname{val} f)$. The first part of the Lemma ensures that $\operatorname{val} L(f) > \psi(\operatorname{val} f)$. Using (21), the latter inequality can be rewritten as

$$\operatorname{val} L(f) > \min_{i \in \{0, \dots, n\}} \operatorname{val}(a_i \phi_\ell^i(f)).$$

Combining the latter inequality with (22), we see that there exist distinct indices $i_1, i_2 \in \{0, \dots, n\}$ such that

$$\operatorname{val}(a_{i_1}\phi_\ell^{i_1}(f)) = \operatorname{val}(a_{i_2}\phi_\ell^{i_2}(f)) = \min_{i\in\{0,\dots,n\}} \operatorname{val}(a_i\phi_\ell^{i}(f)),$$

i.e., such that

$$\operatorname{val} a_{i_1} + \ell^{i_1} \operatorname{val} f = \operatorname{val} a_{i_2} + \ell^{i_2} \operatorname{val} f = \min_{i\in\{0,\dots,n\}} \operatorname{val} a_i + \ell^{i} \operatorname{val} f.$$

Using Lemma 3, we see that $\operatorname{val} f \in -\mathcal{S}(L)$. □

Lemma 12. *For any solution $f \in \mathscr{H}$ of (10), we have*

$$\operatorname{val} f \in \{\pi(\operatorname{val} a_{-\infty})\} \cup -\mathcal{S}(L).$$

Proof. If $f = 0$, then $a_{-\infty} = L(f) = L(0) = 0$ and, hence, $\operatorname{val} f = \operatorname{val} a_{-\infty} = +\infty$. This proves the lemma in this case.

From now on, we assume that $f \neq 0$. We have

$$a_n\phi_\ell^{n}(f) + a_{n-1}\phi_\ell^{n-1}(f) + \cdots + a_0 f - a_{-\infty} = 0. \tag{23}$$

The equality (23) ensures that there exist distinct indices $i_1, i_2 \in \{-\infty, 0, \dots, n\}$ such that

$$\operatorname{val}(a_{i_1}\phi_\ell^{i_1}(f)) = \operatorname{val}(a_{i_2}\phi_\ell^{i_2}(f)) = \min_{i\in\{-\infty,0,\dots,n\}} \operatorname{val}(a_i\phi_\ell^{i}(f)),$$

i.e., such that

$$\operatorname{val} a_{i_1} + \ell^{i_1} \operatorname{val} f = \operatorname{val} a_{i_2} + \ell^{i_2} \operatorname{val} f = \min_{i\in\{-\infty,0,\dots,n\}} \operatorname{val} a_i + \ell^{i} \operatorname{val} f.$$

In what precedes, when $i = -\infty$, by $\operatorname{val}(a_i\phi_\ell^{i}(f))$ and by $\operatorname{val} a_i + \ell^i \operatorname{val} f$, we mean $\operatorname{val} a_{-\infty}$. Using Lemma 3, we see that $\operatorname{val} f \in -\mathcal{S}(L, a_{-\infty})$. We conclude by using the following inclusion that follows directly from the definitions:

$$-\mathcal{S}(L, a_{-\infty}) \subset \{\pi(\operatorname{val} a_{-\infty})\} \cup -\mathcal{S}(L).$$

□

Corollary 13. *Two solutions $f, g \in \mathscr{H}$ of (10) are equal if and only if they are equal on $-\mathcal{S}(L)$.*

Proof. If two solutions f and g of (10) are equal on $-\mathcal{S}(L)$, then $h = g - f$ is equal to 0 on $-\mathcal{S}(L)$ and, hence, $\operatorname{val} h \notin -\mathcal{S}(L)$. But, since h satisfies $L(h) = 0$, Lemma 12 ensures that $\operatorname{val} h \in \{+\infty\} \cup -\mathcal{S}(L)$. Therefore, $\operatorname{val} h = +\infty$ and, hence, $h = 0$, *i.e.*, $f = g$ as claimed. □

Lemma 14. *For any $f \in \mathscr{H}$, we have $\operatorname{supp} L(f) \subset \bigcup_{\gamma\in\operatorname{supp} f} \Psi(\gamma)$.*

Proof. We set, for any $i \in \{0, \dots, n\}$, $a_i = \sum_{j\in\operatorname{supp} a_i} a_{i,j} z^j$. We have

$$L(f) = \sum_{i=0}^{n} a_i\phi_\ell^{i}(f) = \sum_{i=0}^{n} \sum_{j\in\operatorname{supp} a_i} a_{i,j} z^j \phi_\ell^{i}(f),$$

so

$$\begin{aligned}
\operatorname{supp} L(f) &\subset \bigcup_{i=0}^{n} \bigcup_{j\in\operatorname{supp} a_i} \operatorname{supp} z^j \phi_\ell^i(f)\\
&= \bigcup_{i=0}^{n} \bigcup_{j\in\operatorname{supp} a_i} \bigcup_{\gamma\in\operatorname{supp} f} \{\gamma\ell^i + j\}\\
&= \bigcup_{\gamma\in\operatorname{supp} f} \bigcup_{i=0}^{n} \bigcup_{j\in\operatorname{supp} a_i} \{\gamma\ell^i + j\}\\
&= \bigcup_{\gamma\in\operatorname{supp} f} \{\gamma\ell^i + j \mid (\ell^i, j) \in \mathcal{P}(L)\} = \bigcup_{\gamma\in\operatorname{supp} f} \Psi(\gamma).
\end{aligned}$$

□

3.6. **A map of fundamental importance.** The map

$$\begin{array}{rcl} \mathbb{Q} & \to & \{\text{ Finite subsets of } \mathbb{Q}\,\} \\ v & \mapsto & \pi(\Psi(v)) \end{array} \tag{24}$$

will play central role in this paper. For a graphic illustration of this map for the operator L given by (14), we refer to Figure 2.

Let us briefly explain how this map naturally arises when we seek to understand the support of the Hahn series solutions of Mahler equations. Consider $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$ such that $L(f) = 0$. To simplify the presentation, we assume that $\operatorname{supp} f$ has at least two elements and we ask: what are the possible values for the two least elements $\gamma_0 < \gamma_1$ of $\operatorname{supp} f$? We have $\gamma_0 = \operatorname{val} f$, $\gamma_1 = \operatorname{val} f_{|\mathbb{Q}_{>\gamma_0}}$ and

$$f = f_{|\mathbb{Q}_{>\gamma_1}} + f_{\gamma_1} z^{\gamma_1} + f_{\gamma_0} z^{\gamma_0}.$$

What are the possible values for γ_0? Lemma 12 provides an immediate answer to this question: $\gamma_0 \in -\mathcal{S}(L)$.

What are the possible values for γ_1? In order to answer this question, note that the equations $L(f) = 0$ and $f = f_{|\mathbb{Q}_{>\gamma_0}} + f_{\gamma_0} z^{\gamma_0}$ imply $L(f_{|\mathbb{Q}_{>\gamma_0}}) = -f_{\gamma_0} L(z^{\gamma_0})$. Lemma 12 ensures that $\gamma_1 = \operatorname{val} f_{|\mathbb{Q}_{>\gamma_0}} \in \{\pi(\operatorname{val} L(z^{\gamma_0}))\} \cup -\mathcal{S}(L)$. But, it follows from Lemma 14 that $\operatorname{supp} L(z^{\gamma_0}) \subset \Psi(\gamma_0)$. So, $\gamma_1 \in \pi(\Psi(\gamma_0)) \cup -\mathcal{S}(L)$. Since $\gamma_0 \in -\mathcal{S}(L)$ and since $-\mathcal{S}(L) \subset \bigcup_{v\in-\mathcal{S}(L)} \pi(\Psi(v))$ as a consequence of the first assertion of Lemma 15 below, we get

$$\gamma_1 \in \bigcup_{v\in-\mathcal{S}(L)} \pi(\Psi(v)).$$

We see the map (24) naturally appear here. Iterating this in order to reach more and more elements of the support of f, we can guess that the map (24) and its iterates should play a central role in the study of the support of the Hahn series solution of (10). We will see in section 4 and, more precisely, in Theorem 17 that this is indeed the case.

Note the following result for further use.

Lemma 15. *For all $v \in \mathbb{Q}$, we have*

$$v \in \pi(\Psi(v))$$

and

$$\min \pi(\Psi(v)) = \pi(\min \Psi(v)) = \pi(\psi(v)) = v. \tag{25}$$

Proof. In order to prove the lemma, it is sufficient to prove (25). The first equality in (25) follows from the fact that π is increasing by Lemma 7. The second equality in (25) follows from the definition of ψ. The third equality in (25) follows form the fact that π and ψ are inverse of each other by Lemma 7. □

4. A receptacle $\mathcal{V}$ for the support of the solutions

Throughout this section, we consider a Mahler operator

$$L = a_n \phi_\ell^n + a_{n-1}\phi_\ell^{n-1} + \cdots + a_0 \tag{26}$$

with coefficients $a_0, \ldots, a_n \in \mathbf{K}[z]$ such that $a_0 a_n \neq 0$. We let

$$\mathrm{Sol}(L, \mathscr{H}) = \{f \in \mathscr{H} \mid L(f) = 0\}$$

be the $\mathbf{K}$-vector space of solutions of L in $\mathscr{H}$. In what follows, we will use the following terminology:

Definition 16. *We say that a subset $\mathcal{Q}$ of $\mathbb{Q}$ is computable if there exists an algorithm which takes a rational number q as input and returns whether it belongs to $\mathcal{Q}$ or not.*

The aim of this section is to describe a computable well-ordered subset $\mathcal{V}$ of $\mathbb{Q}$ containing the support of any Hahn series solution of L, *i.e.*, such that

$$\mathrm{Sol}(L, \mathscr{H}) \subset \mathscr{H}_{|\mathcal{V}} \tag{27}$$

(and satisfying a technical but important stability condition with respect to the map (24)).

Theorem 17. *Let $(\mathcal{V}_i)_{i\geq 0}$ be the sequence of finite subsets of $\mathbb{Q}$ defined as follows:*

- $\mathcal{V}_0 = -\mathcal{S}(L)$;
- $\forall i \geq 0,\ \mathcal{V}_{i+1} = \bigcup_{v\in\mathcal{V}_i} \pi(\Psi(v))$.

The sequence $(\mathcal{V}_i)_{i\geq 0}$ is nondecreasing and the set $\mathcal{V} = \bigcup_{i\geq 0} \mathcal{V}_i$ has the following properties:

(1) $\mathrm{Sol}(L, \mathscr{H}) \subset \mathscr{H}_{|\mathcal{V}}$;
(2) *$\mathcal{V}$ is well-ordered;*
(3) *$\mathcal{V}$ is computable;*
(4) $-\mathcal{S}(L) \subset \mathcal{V}$;
(5) $\bigcup_{v\in\mathcal{V}} \pi(\Psi(v)) = \mathcal{V}$.

Property (4) is obvious and Property (5) follows immediately from Lemma 15 and the construction of $\mathcal{V}$. They will be freely used in the proofs of the other assertions of Theorem 17 given in the next subsections. The fact that $(\mathcal{V}_i)_{i\geq 0}$ is nondecreasing is proved in section 4.1. Section 4.2 gives a couple of basic properties of $\mathcal{V}$ that will be used for the proofs of properties (1), (2) and (3) of Theorem 17 but also latter in the paper. Properties (1), (2) and (3) of Theorem 17 are proved in sections 4.3, 4.4 and 4.5 respectively.

Remark 18. *The existence of a subset $\mathcal{V}$ of $\mathbb{Q}$ satisfying properties (1) and (2) is obvious: the support of any element of* $\mathrm{Sol}(L,\mathscr{H})$ *is included in the union of the supports of the elements of an arbitrary basis of* $\mathrm{Sol}(L,\mathscr{H})$ *(which is a finite dimensional sub-*$\mathbf{K}$*-vector space of $\mathscr{H}$ with dimension at most n). But, this is not sufficient for our purpose, all the properties listed in Theorem 17 will be used. In particular:*

- *property (3) is of fundamental importance for the algorithmic considerations of this paper; it is one of the crucial ingredients that makes our answer to Question 1, given by Theorem 62, possible;*
- *property (4) is a precaution to ensure that property (1) is satisfied;*
- *property (5) is essential in section 6 to build the set $\mathcal{R}$ mentioned in the introduction.*

Example 19. The sets $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2$ are computed in section 9.3 for the operator L given by (14).

In what follows, we will use the following notations:

- we continue with the notations $\mu_1, \dots, \mu_K$, α_k, β_k, *etc*, from section 3;
- we let $d \in \mathbb{Z}_{\geq 1}$ be a common multiple of the denominators of the slopes $\mu_1, \dots, \mu_K$;
- for any subset $\mathcal{Q}$ of $\mathbb{Q}$ and any $\gamma \in \mathbb{Q}$, we set $\mathcal{Q}_{>\gamma} = \mathcal{Q} \cap \mathbb{Q}_{>\gamma}$, $\mathcal{Q}_{\geq\gamma} = \mathcal{Q} \cap \mathbb{Q}_{\geq\gamma}$, $\mathcal{Q}_{<\gamma} = \mathcal{Q} \cap \mathbb{Q}_{<\gamma}$ and $\mathcal{Q}_{\leq\gamma} = \mathcal{Q} \cap \mathbb{Q}_{\leq\gamma}$.

4.1. Proof of the fact that $(\mathcal{V}_i)_{i\geq 0}$ is nondecreasing in Theorem 17.

Consider $i \in \mathbb{Z}_{\geq 0}$ and $v \in \mathcal{V}_i$. Lemma 15 ensures that $v \in \pi(\Psi(v))$. But, $\pi(\Psi(v)) \subset \pi(\bigcup_{w\in\mathcal{V}_i} \Psi(w)) = \mathcal{V}_{i+1}$. So, $v \in \mathcal{V}_{i+1}$. This shows that $\mathcal{V}_i \subset \mathcal{V}_{i+1}$.

4.2. Basic properties of $\mathcal{V}$.

Lemma 20. *The set $\mathcal{V}$ has a minimal element given by* $\min \mathcal{V} = \min \mathcal{V}_0 = \min -\mathcal{S}(L) = -\mu_K$.

Proof. We claim that, for all $i \in \mathbb{Z}_{\geq 0}$,

$$\min \mathcal{V}_{i+1} = \min \mathcal{V}_i. \tag{28}$$

Indeed, for all $i \in \mathbb{Z}_{\geq 0}$, we have $\mathcal{V}_{i+1} = \pi(\bigcup_{v\in\mathcal{V}_i} \Psi(v)) = \bigcup_{v\in\mathcal{V}_i} \pi(\Psi(v))$. But, Lemma 15 ensures that, for all $v \in \mathbb{Q}$, $\min \pi(\Psi(v)) = v$. So,

$$\min \mathcal{V}_{i+1} = \min \bigcup_{v\in\mathcal{V}_i} \pi(\Psi(v)) = \min_{v\in\mathcal{V}_i} \min \pi(\Psi(v)) = \min_{v\in\mathcal{V}_i} v = \min \mathcal{V}_i,$$

whence our claim. Now, the equality $\min \mathcal{V} = \min \mathcal{V}_0$ follows clearly from the fact that $\mathcal{V} = \bigcup_{i\geq 0} \mathcal{V}_i$ and from (28). The equality $\min \mathcal{V}_0 = \min -\mathcal{S}(L)$ follows from the fact that $\mathcal{V}_0 = -\mathcal{S}(L)$ by definition. The equality $\min -\mathcal{S}(L) = -\mu_K$ follows from the fact that μ_K is the greatest slope of L. □

Lemma 21. *Consider $v \in \mathcal{V}$. Let M be the least element of $\mathbb{Z}_{\geq 0}$ such that $v \in \mathcal{V}_M$. There exist $v_0 < \cdots < v_{M-1} < v_M = v$ in $\mathcal{V}$ such that*

- *$v_0 \in -\mathcal{S}(L)$;*
- *$v_{i+1} \in \pi\left(\Psi(v_i)\right)$ for any $i \in \{0, \dots, M-1\}$.*

Proof. The case $M = 0$ being obvious, we will assume in the remainder of this proof that $M \geq 1$. It follows immediately from the definition of the $\mathcal{V}_i$ that there exist $v_0, \dots, v_{M-1}, v_M = v$ in $\mathcal{V}$ such that

- $v_0 \in -\mathcal{S}(L)$;
- $v_{i+1} \in \pi(\Psi(v_i))$ for any $i \in \{0, \ldots, M-1\}$.

Lemma 15 guarantees that, for all $i \in \{0, \ldots, M-1\}$, $\min \pi(\Psi(v_i)) = v_i$, so $v_{i+1} \geq v_i$. Therefore, we have $v_0 \leq \cdots \leq v_M$. If one these inequalities were an equality, then v_M would belong to $\mathcal{V}_{M-1}$ and this would contradict the minimality of M. Thus, $v_0 < \cdots < v_M$. This concludes the proof. □

4.3. **Proof of** (1) **of Theorem 17.** Consider $f \in \mathrm{Sol}(L, \mathscr{H})$. We want to prove that $\operatorname{supp} f \subset \mathcal{V}$. Assume on the contrary that $\operatorname{supp} f \not\subset \mathcal{V}$. Then, the set $\operatorname{supp}(f) \setminus \mathcal{V}$ is nonempty and well-ordered. In particular, we can consider

$$\gamma_{\min} = \min\left(\operatorname{supp}(f) \setminus \mathcal{V}\right).$$

By minimality of $\gamma_{\min}$, we have

$$\operatorname{supp} f_{|\mathbb{Q}_{<\gamma_{\min}}} \subset \mathcal{V}. \tag{29}$$

Consider the decomposition

$$f = f_{|\mathbb{Q}_{<\gamma_{\min}}} + f_{|\mathbb{Q}_{\geq\gamma_{\min}}}.$$

The equality $L(f) = 0$ implies the equality $L(f_{|\mathbb{Q}_{\geq\gamma_{\min}}}) = -L(f_{|\mathbb{Q}_{<\gamma_{\min}}})$ and we infer from Lemma 12 that

$$\gamma_{\min} = \operatorname{val} f_{|\mathbb{Q}_{\geq\gamma_{\min}}} \in \{\pi(\operatorname{val} L(f_{|\mathbb{Q}_{<\gamma_{\min}}}))\} \cup -\mathcal{S}(L). \tag{30}$$

Since $\gamma_{\min} \in \operatorname{supp}(f) \setminus \mathcal{V}$ and $-\mathcal{S}(L) \subset \mathcal{V}$, we have $\gamma_{\min} \notin -\mathcal{S}(L)$ so (30) gives

$$\gamma_{\min} = \pi(\operatorname{val} L(f_{|\mathbb{Q}_{<\gamma_{\min}}})). \tag{31}$$

In particular, this implies that $L(f_{|\mathbb{Q}_{<\gamma_{\min}}}) \neq 0$. Lemma 14 ensures that

$$\operatorname{val} L(f_{|\mathbb{Q}_{<\gamma_{\min}}}) \in \operatorname{supp} L(f_{|\mathbb{Q}_{<\gamma_{\min}}}) \subset \bigcup_{\gamma \in \operatorname{supp} f_{|\mathbb{Q}_{<\gamma_{\min}}}} \Psi(\gamma). \tag{32}$$

Combining (31) and (32), we get

$$\gamma_{\min} \in \pi\left(\bigcup_{\gamma \in \operatorname{supp} f_{|\mathbb{Q}_{<\gamma_{\min}}}} \Psi(\gamma)\right). \tag{33}$$

The right-hand side of (33) is included in $\pi(\bigcup_{\gamma\in\mathcal{V}} \Psi(\gamma))$ by (29) and the latter set is equal to $\mathcal{V}$ by property (5) of Theorem 17. So, $\gamma_{\min} \in \mathcal{V}$. This is a contradiction.

4.4. **Proof of** (2) **of Theorem 17.** We argue by contradiction: we assume that $\mathcal{V}$ is not well-ordered. Thus, the set D made up of the infinite decreasing sequences with values in $\mathcal{V}$ is nonempty. Lemma 20 ensures that $\mathcal{V} \subset \mathbb{Q}_{\geq -\mu_K}$. Therefore, any element of D has a limit in $\mathbb{R}_{\geq -\mu_K}$. We let $E \subset \mathbb{R}_{\geq -\mu_K}$ be the (nonempty) set made of these limits. We set $w = \inf E \in \mathbb{R}_{\geq -\mu_K}$. It is easily seen that $w \in E$. We let $(w_m)_{m\geq 0}$ be an arbitrary element of D tending to w.

Note that, for all $m \geq 0$, $w_m > w_{m+1} \geq w$, so $w_m > w$. Consider $k \in \{1, \ldots, K\}$ such that $-\mu_k \leq w < -\mu_{k-1}$ with the convention $\mu_0 = -\infty$. Since $(w_m)_{m\geq 0}$ is decreasing and tends to w, up to replacing $(w_m)_{m\ \geq 0}$ by $(w_{m+M})_{m\geq 0}$ with $M \in \mathbb{Z}_{\geq 0}$ large enough, we can assume that, for all

$m \geq 0$, $-\mu_k < w_m < -\mu_{k-1}$, where the first inequality is legitimate because $w_m > w \geq -\mu_k$.

Lemma 21 ensures that, for any $M \geq 0$, there exist $r_M \in \mathbb{Z}_{\geq 0}$ and $v_{M,0} < \cdots < v_{M,r_M} = w_M$ in $\mathcal{V}$ such that

- $v_{M,0} \in -\mathcal{S}(L)$;
- $v_{M,i+1} \in \pi\ (\Psi(v_{M,i}))$ for any $i \in \{0, \ldots, r_M - 1\}$.

For any $M \geq 0$, we have $v_{M,r_M} = w_M > w$, so one can consider the least element r'_M of $\{0, \ldots, r_M\}$ such that $v_{M,r'_M} > w$. It is important to notice that $r'_M \geq 1$ for all $M \geq 0$, because $-\mu_k \leq w < v_{M,r'_M} \leq w_M < -\mu_{k-1}$, so $v_{M,r'_M} \notin -\mathcal{S}(L)$ and, in particular, $v_{M,r'_M} \neq v_{M,0}$. The inequality $w_M \geq v_{M,r'_M} > w$ and the fact that $(w_m)_{m\geq 0}$ tends to w show that $(v_{M,r'_M})_{M\geq 0}$ tends to w and that, up to extracting a subsequence, we can assume that $(v_{M,r'_M})_{M\geq 0}$ is decreasing.

We set $q_M = \psi(v_{M,r'_M})$, so that $q_M \in \Psi(v_{M,r'_M})$ is such that $v_{M,r'_M} = \pi(q_M)$ by Lemma 7. As $-\mu_k \leq w < v_{M,r'_M} = \pi(q_M) \leq w_M < -\mu_{k-1}$, it follows from Lemma 10 that, for all $M \geq 0$,

$$v_{M,r'_M} = \pi(q_M) = \frac{q_M - \beta_{k-1}}{\ell^{\alpha_{k-1}}} \tag{34}$$

where $(\ell^{\alpha_{k-1}}, \beta_{k-1}) \in \mathcal{P}(L)$ is the left endpoint of the edge of $\mathcal{N}(L)$ with slope μ_k. Moreover, since, for all $M \geq 0$, $v_{M,r'_M} \in \pi(\Psi(v_{M,r'_M-1}))$ and since ψ and π are inverse of each other by Lemma 7, we have $q_M = \psi(v_{M,r'_M}) \in \psi(\pi(\Psi(v_{M,r'_M-1}))) = \Psi(v_{M,r'_M-1})$ and, hence, there exists $(\ell^{i_M}, j_M) \in \mathcal{P}(L)$ such that

$$q_M = \ell^{i_M} v_{M,r'_M-1} + j_M. \tag{35}$$

Since $\mathcal{P}(L)$ is finite, up to extracting a subsequence, we can assume that $i_M = \alpha$ and $j_M = \beta$ do not depend on $M \geq 0$. Combining (34) and (35), we obtain, for all $M \geq 0$,

$$v_{M,r'_M} = \frac{\ell^{\alpha} v_{M,r'_M-1} + \beta - \beta_{k-1}}{\ell^{\alpha_{k-1}}}$$

and, hence,

$$v_{M,r'_M-1} = \frac{\ell^{\alpha_{k-1}} v_{M,r'_M} + \beta_{k-1} - \beta}{\ell^{\alpha}}.$$

The latter equality can be rewritten as $v_{M,r'_M-1} = \delta(v_{M,r'_M})$ where $\delta : \mathbb{R} \to \mathbb{R}$ is the increasing affine function defined by $\delta(x) = \frac{\ell^{\alpha_{k-1}}x + \beta_{k-1} - \beta}{\ell^{\alpha}}$. Since $(v_{M,r'_M})_{M\geq 0}$ is decreasing, this implies that $(v_{M,r'_M-1})_{M\geq 0}$ is decreasing as well and, hence, belongs to D. Let w' be the limit of $(v_{M,r'_M-1})_{M\geq 0}$. Since $(v_{M,r'_M-1})_{M\geq 0}$ is decreasing and satisfies, for all $M \geq 0$, $v_{M,r'_M-1} \leq w$, we have $w' < w$. This contradicts the minimality of w and concludes the proof of (2) of Theorem 17.

4.5. **Proof of (3) of Theorem 17.** We set

$$\mathbb{Z}_{d,\ell} = \bigcup_{i \in \mathbb{Z}_{\geq 0}} \frac{1}{d\ell^i}\mathbb{Z} \subset \mathbb{Q}.$$

Note that we do not require the integers d and ℓ to be coprime. We introduce the following two maps that will play an important role in this section:

$$\begin{array}{rcl} h : \mathbb{Z}_{d,\ell} & \to & \mathbb{Z}_{\geq 0} \\ v & \mapsto & h(v) = \min\left\{i \in \mathbb{Z}_{\geq 0} \mid v \in \frac{1}{d\ell^i}\mathbb{Z}\right\} \end{array}$$

and

$$\begin{array}{rcl} \epsilon : \mathbb{Q} & \to & \mathbb{Q}_{>0} \cup \{+\infty\} \\ v & \mapsto & \epsilon(v) = \min \mathcal{V}_{>v} - v \end{array} \tag{36}$$

with the convention $\epsilon(v) = +\infty$ if $\mathcal{V}_{>v} = \emptyset$. The fact that ϵ is well-defined and takes its values in $\mathbb{Q}_{>0} \cup \{+\infty\}$, *i.e.*, the fact that $\mathcal{V}_{>v}$ has a minimal element if it is not empty, follows from the fact that $\mathcal{V}$ is well-ordered.

4.5.1. *Computability of $\mathcal{V}$: description of an algorithm.* Let us temporarily admit the following result which will be proved in section 4.5.2.

Proposition 22. *For any $v \in \mathbb{Q}$, the following properties are equivalent:*

(1) $v \in \mathcal{V}$;

(2) $v \in \mathbb{Z}_{d,\ell}$, $v \geq -\mu_K$ and $v \in \mathcal{V}_{\iota(v)}$ where

$$\iota(v) = \left\lfloor (n+1)\frac{v+\mu_K}{\tau} + h(v) \right\rfloor \tag{37}$$

with

$$\tau := \min\left\{\epsilon(-\mu_1), \ldots, \epsilon(-\mu_K), (d\ell^n)^{-1}\right\} \in \mathbb{Q}_{>0}. \tag{38}$$

The interest of this result lies in the fact that, while $\mathcal{V}$ is infinite, the set $\mathcal{V}_{\iota(v)}$ is finite and can be easily computed (directly from its definition) once $\iota(v)$ has been computed. Unfortunately, $\iota(v)$ is not easy to compute. More precisely, while in formula (37) the values of n, μ_K and $h(v)$ can be easily computed, the value of τ cannot. This problem can be solved using section 5 below, where an algorithm for computing a positive lower bound $\breve{\tau}$ on τ is provided. Whence an upper bound

$$\breve{\iota}(v) = \left\lfloor (n+1)\frac{v+\mu_K}{\breve{\tau}} + h(v) \right\rfloor$$

on $\iota(v)$ for any $v \in \mathbb{Z}_{d,\ell}$ such that $v \geq -\mu_K$. Now, since $(\mathcal{V}_i)_{i\geq 0}$ is nondecreasing, we have $\mathcal{V}_{\iota(v)} \subset \mathcal{V}_{\breve{\iota}(v)}$ and Proposition 22 implies that the following properties are equivalent:

(1) $v \in \mathcal{V}$;

(2) $v \in \mathbb{Z}_{d,\ell}$, $v \geq -\mu_K$ and $v \in \mathcal{V}_{\breve{\iota}(v)}$.

This leads to the following algorithm, which uses Algorithm 44 described at the end of section 5 for computing a positive lower bound $\breve{\tau}$ on τ.

Algorithm 23.

Input: L a Mahler operator with coefficients in $\mathbf{K}[z]$, $v \in \mathbb{Q}$.
Output: whether or not v is in $\mathcal{V}$.

---

if $v \notin \mathbb{Z}_{d,\ell}$ or $v < -\mu_K$ **then**

return "v is not in $\mathcal{V}$"
otherwise
compute n, μ_K and $h(v)$
compute a positive lower bound $\breve{\tau}$ on τ using Algorithm 44
compute $\breve{\imath} = \lfloor (n+1)\frac{v+\mu_K}{\breve{\tau}} + h(v) \rfloor$
compute $\mathcal{V}_{\breve{\imath}}$
if $v \in \mathcal{V}_{\breve{\imath}}$ **then**
return "v is in $\mathcal{V}$"
otherwise
return "v is not in $\mathcal{V}$"
end if
end if

4.5.2. *Proof of Proposition 22.* Proposition 22 is proved at the very end of this section, after a series of lemmas.

Lemma 24. *We have*

$$\mathcal{V} \subset \mathbb{Z}_{d,\ell}. \tag{39}$$

Moreover, for any $v \in \mathbb{Z}_{d,\ell}$ and $w \in \pi(\Psi(v))$, we have $w \in \mathbb{Z}_{d,\ell}$ and

$$h(v) \leq h(w) + n. \tag{40}$$

Proof. Before proving (39), note that

- $\mathcal{V}_0 \subset \mathbb{Z}_{d,\ell}$;
- for any $v \in \mathbb{Z}_{d,\ell}$, $\pi(\Psi(v)) \subset \mathbb{Z}_{d,\ell}$.

Indeed, the first property holds because d is a common denominator of the elements of $\mathcal{V}_0 = -\mathcal{S}(L)$. The second property holds because, by definition of π and Ψ, for any $v \in \mathbb{Z}_{d,\ell}$ and any $w \in \pi(\Psi(v))$, there exist $(\ell^i, j), (\ell^{i'}, j') \in \mathcal{P}(L)$ such that

$$w = \frac{\ell^i v + j - j'}{\ell^{i'}} \in \frac{1}{\ell^{i'}}\mathbb{Z}_{d,\ell} = \mathbb{Z}_{d,\ell}.$$

Let us now prove (39). Since $\mathcal{V} = \cup_{i\geq 0}\mathcal{V}_i$, it is equivalent to prove that, for all $i \in \mathbb{Z}_{\geq 0}$, $\mathcal{V}_i \subset \mathbb{Z}_{d,\ell}$. The latter property can be proved by induction on $i \in \mathbb{Z}_{\geq 0}$. Indeed, the two properties noticed at the beginning of the proof show that $\mathcal{V}_0 \subset \mathbb{Z}_{d,\ell}$ and that if, for a given $i \in \mathbb{Z}_{\geq 0}$, we have $\mathcal{V}_i \subset \mathbb{Z}_{d,\ell}$, then we have $\mathcal{V}_{i+1} = \bigcup_{v\in\mathcal{V}_i} \pi(\Psi(v)) \subset \mathbb{Z}_{d,\ell}$.

Let us now prove (40). By definition of $h(w)$, we have $w \in \frac{1}{d\ell^{h(w)}}\mathbb{Z}$, so

$$v = \frac{\ell^{i'} w + j' - j}{\ell^i} \in \frac{1}{\ell^i}\frac{1}{d\ell^{h(w)}}\mathbb{Z} = \frac{1}{d\ell^{h(w)+i}}\mathbb{Z}.$$

Thus, we have $h(v) \leq h(w) + i \leq h(w) + n$ and this proves (40). □

Remark 25. *Consider two multiplicatively independent integers $\ell_1, \ell_2 \geq 2$. In [AB17], Adamczewski and Bell give a proof of a conjecture of Loxton and van der Poorten asserting that any Puiseux series solution of both a ℓ_1-Mahler equation and of a ℓ_2-Mahler equation belongs to $\bigcup_{d\geq 1}\mathbb{C}(z^{1/d})$. As mentioned in the Introduction, an alternative proof was given later by Shäfke and Singer in [SS19]. The first part of Lemma 24 can be used to extend this* *result to Hahn series. Indeed, let $f \in \mathscr{H}$ be solution of both a ℓ_1-Mahler equation and of a ℓ_2-Mahler equation with coefficients in $\mathbb{C}(z)$. Then, it follows from Lemma 24 that* $\operatorname{supp} f$ *is included in $\mathbb{Z}_{d_1,\ell_1}$ and in $\mathbb{Z}_{d_2,\ell_2}$ for some integers $d_1, d_2 \geq 1$. But, $\mathbb{Z}_{d_1,\ell_1} \cap \mathbb{Z}_{d_2,\ell_2} \subset \frac{1}{d}\mathbb{Z}$ for some integer $d \geq 1$ because ℓ_1 and ℓ_2 are multiplicatively independent. Thus, f is a Puiseux series solution of both a ℓ_1-Mahler equation and a ℓ_2-Mahler equation and, hence, $f \in \bigcup_{d\geq 1} \mathbb{C}(z^{1/d})$.*

Lemma 26. *Let $v \in \mathbb{Z}_{d,\ell}$ and $w \in \pi(\Psi(v))$ satisfy*

$$-\mu_k + \breve{\epsilon}_k \leq v < w < -\mu_{k-1} \tag{41}$$

for some $k \in \{1, \ldots, K\}$ and some lower bound $\breve{\epsilon}_k > 0$ on $\epsilon(-\mu_k)$. Then, at least one of the following properties holds:

- *$w \geq v + \breve{\epsilon}_k$;*
- *$w \geq v + \frac{1}{d\ell^n}$;*
- *$h(w) > h(v)$.*

Proof. By definition of Ψ, there exists $(\ell^\alpha, \beta) \in \mathcal{P}(L)$ such that $w = \pi(\ell^\alpha v + \beta)$. Since $-\mu_k \leq w = \pi(\ell^\alpha v + \beta) < -\mu_{k-1}$, Lemma 10 ensures that

$$w = \pi(\ell^\alpha v + \beta) = \frac{\ell^\alpha v + \beta - \beta_{k-1}}{\ell^{\alpha_{k-1}}}. \tag{42}$$

We now distinguish the cases $\alpha = \alpha_{k-1}$, $\alpha > \alpha_{k-1}$ and $\alpha < \alpha_{k-1}$.

Case $\alpha = \alpha_{k-1}$. In this case, (42) can be rewritten as $w = v + \frac{\beta - \beta_{k-1}}{\ell^{\alpha_{k-1}}}$. Since $w > v$, we have $\beta - \beta_{k-1} > 0$. Since β and β_{k-1} belong to $\mathbb{Z}$, we have $\beta - \beta_{k-1} \geq 1$. It follows that

$$w \geq v + \frac{1}{\ell^{\alpha_{k-1}}} \geq v + \frac{1}{\ell^n} \geq v + \frac{1}{d\ell^n}$$

and the lemma holds in this case.

Case $\alpha > \alpha_{k-1}$. In this case, since $(\ell^{\alpha_{k-1}}, \beta_{k-1})$ is the left endpoint of the edge of $\mathcal{N}(L)$ of slope μ_k, the slope of the vector joining $(\ell^{\alpha_{k-1}}, \beta_{k-1})$ to (ℓ^α, β) is greater than or equal to μ_k, that is,

$$\frac{\beta - \beta_{k-1}}{\ell^\alpha - \ell^{\alpha_{k-1}}} \geq \mu_k.$$

So, we have

$$\begin{aligned} w - v = \frac{(\ell^\alpha - \ell^{\alpha_{k-1}})v + \beta - \beta_{k-1}}{\ell^{\alpha_{k-1}}} &\geq \frac{(\ell^\alpha - \ell^{\alpha_{k-1}})v + (\ell^\alpha - \ell^{\alpha_{k-1}})\mu_k}{\ell^{\alpha_{k-1}}} \\ &= \frac{(\ell^\alpha - \ell^{\alpha_{k-1}})(v + \mu_k)}{\ell^{\alpha_{k-1}}} \geq \frac{(\ell^\alpha - \ell^{\alpha_{k-1}})\breve{\epsilon}_k}{\ell^{\alpha_{k-1}}} \geq \breve{\epsilon}_k \end{aligned}$$

and the lemma holds in this case.

Case $\alpha < \alpha_{k-1}$. Write $v = \frac{M}{d\ell^{h(v)}}$ with $M \in \mathbb{Z}$. Equation (42) can be rewritten as

$$w = \frac{M}{d\ell^{h(v)+\alpha_{k-1}-\alpha}} + \frac{\beta - \beta_{k-1}}{\ell^{\alpha_{k-1}}}. \tag{43}$$

We now distinguish two cases:

- if $h(v) - \alpha > 0$, then equation (43) shows that $h(w) = h(v) + \alpha_{k-1} - \alpha > h(v)$ and the lemma holds;

- if $h(v) \leq \alpha$, then equation (43) shows that $h(w) \leq \alpha_{k-1}$; so v and w both belong to $\frac{1}{d\ell^{\alpha_{k-1}}}\mathbb{Z}$; it follows that $w-v$ is a positive element of $\frac{1}{d\ell^{\alpha_{k-1}}}\mathbb{Z}$, so $w-v \geq \frac{1}{d\ell^{\alpha_{k-1}}} \geq \frac{1}{d\ell^{n}}$ and the lemma holds.

□

Lemma 27. *Consider $k \in \{1,\dots,K\}$ and let $\breve{\epsilon}_k > 0$ be a lower bound on $\epsilon(-\mu_k)$. Consider $M \in \mathbb{Z}_{\geq 0}$ and $v_0,\dots,v_M \in \mathbb{Z}_{d,\ell}$ such that*

- *$\forall i \in \{0,\dots,M-1\}$, $v_{i+1} \in \pi(\Psi(v_i))$;*
- *$-\mu_k + \breve{\epsilon}_k \leq v_0 < \cdots < v_{M-1} < v_M < -\mu_{k-1}$.*

Then, we have

$$M < (n+1)\frac{v_M + \mu_k}{\min\{\breve{\epsilon}_k, (d\ell^n)^{-1}\}} + h(v_M).$$

Proof. Set

$$m_k = \min\left\{\breve{\epsilon}_k, (d\ell^n)^{-1}\right\}.$$

Consider

$$E_1 = \{i \in \{0,\dots,M-1\} \mid h(v_i) \geq h(v_{i+1})\} \tag{44}$$

and

$$E_2 = \{i \in \{0,\dots,M-1\} \mid h(v_i) < h(v_{i+1})\}. \tag{45}$$

Set $M_1 = \sharp E_1$ and $M_2 = \sharp E_2$. Since $\{E_1, E_2\}$ is a partition of $\{0,\dots,M-1\}$, we have

$$M = M_1 + M_2.$$

We have :

- $\forall i \in \{0,\dots,M-1\}$, $h(v_i) \leq h(v_{i+1}) + n$ by Lemma 24;
- if $i \in E_2$, then $h(v_i) < h(v_{i+1})$ and, hence, $h(v_i) \leq h(v_{i+1}) - 1$ because $h(v_i)$ and $h(v_{i+1})$ are integers.

It follows that

$$h(v_0) \leq h(v_M) + nM_1 - M_2.$$

Since $h(v_0) \geq 0$, we get

$$M_2 \leq h(v_M) + nM_1. \tag{46}$$

We shall now give an upper bound on M_1. On the one hand, since $-\mu_k + \breve{\epsilon}_k \leq v_0 < \cdots < v_{M-1} < v_M$, we have

$$\sum_{i \in E_1} v_{i+1} - v_i \leq \sum_{i=0}^{M-1} v_{i+1} - v_i = v_M - v_0 \leq v_M + \mu_k - \breve{\epsilon}_k < v_M + \mu_k. \tag{47}$$

On the other hand, for any $i \in E_1$, we have $h(v_i) \geq h(v_{i+1})$. Since $-\mu_k + \breve{\epsilon}_k < v_i < v_{i+1} < -\mu_{k-1}$, Lemma 26 ensures that $v_i \leq v_{i+1} - m_k$. It follows that

$$\sum_{i \in E_1} v_{i+1} - v_i \geq \sum_{i \in E_1} m_k = M_1 m_k. \tag{48}$$

Combining (47) and (48), we get

$$M_1 < \frac{v_M + \mu_k}{m_k}. \tag{49}$$

Finally, combining (46) and (49), we obtain

$$M = M_1 + M_2 \leq h(v_M) + (n+1)M_1 < h(v_M) + (n+1)\frac{v_M + \mu_k}{m_k}.$$

$\square$

Lemma 28. *Consider $M \in \mathbb{Z}_{\geq 0}$ and $v_0, \ldots, v_M \in \mathcal{V}$ such that*

- $v_0 < \cdots < v_{M-1} < v_M$*;*
- $\forall i \in \{0, \ldots, M-1\}$*,* $v_{i+1} \in \pi(\Psi(v_i))$*.*

Then,

$$M \leq (n+1)\frac{v_M + \mu_K}{\tau} + h(v_M)$$

where

$$\tau = \min\left\{\epsilon(-\mu_1), \ldots, \epsilon(-\mu_K), (d\ell^n)^{-1}\right\} > 0.$$

Proof. As in the proof of Lemma 27, we consider the sets E_1 and E_2 defined by formulas (44) and (45) respectively, we set $M_1 = \sharp E_1$ and $M_2 = \sharp E_2$ and we have

$$M = M_1 + M_2$$

and

$$M_2 \leq h(v_M) + nM_1.$$

As in the proof of Lemma 27, we shall now give an upper bound on M_1. We first claim that, for all $i \in E_1$, we have

$$v_i \leq v_{i+1} - \tau. \tag{50}$$

In order to prove this claim, let us first recall that $\mathcal{V} \subset \mathbb{Q}_{\geq -\mu_K}$ by Lemma 20, so one of the following cases is satisfied.

Case 1: there exists $k \in \{1, \ldots, K\}$ such that $-\mu_k \leq v_i < v_{i+1} < -\mu_{k-1}$. We distinguish the following two subcases.

Subcase 1.1: $-\mu_k + \tau \leq v_i$. Since $i \in E_1$, we have $h(v_i) \geq h(v_{i+1})$. Moreover, we have $-\mu_k + \tau \leq v_i < v_{i+1} < -\mu_{k-1}$ by hypothesis. So, Lemma 26 ensures that $v_i \leq v_{i+1} - \tau$ as claimed.

Subcase 1.2: $v_i < -\mu_k + \tau$. In this case, since $v_i \in \mathcal{V}$, the definition of $\epsilon(-\mu_k)$ ensures that $v_i = -\mu_k$. Since $v_{i+1} \in \mathcal{V}$, we have $v_i = -\mu_k \leq v_{i+1} - \tau$ as claimed.

Case 2: there exists $k \in \{1, \ldots, K-1\}$ such that $v_i < -\mu_k \leq v_{i+1}$. We distinguish the following two subcases.

Subcase 2.1: $v_{i+1} = -\mu_k$. In this case, we have $v_{i+1} \in \frac{1}{d}\mathbb{Z}$, so $h(v_{i+1}) = 0$. Lemma 24 implies $h(v_i) \leq n$, *i.e.*, $v_i \in \frac{1}{d\ell^n}\mathbb{Z}$. Therefore, v_i and v_{i+1} are elements of $\frac{1}{d\ell^n}\mathbb{Z}$ such that $v_i < v_{i+1}$, so we have $v_i \leq v_{i+1} - \frac{1}{d\ell^n} \leq v_{i+1} - \tau$ as claimed.

Subcase 2.2: $v_{i+1} > -\mu_k$. Since $v_{i+1} \in \mathcal{V}$, the definition of $\epsilon(-\mu_k)$ ensures that $v_{i+1} \geq -\mu_k + \tau$. But, by hypothesis, $-\mu_k > v_i$. So, $v_{i+1} > v_i + \tau$ and our claim is proved in this case as well.

Now that the inequality (50) is justified, we can argue as we did in Lemma 27 for proving (49) in order to prove that

$$M_1 \leq \frac{v_M + \mu_K}{\tau}.$$

Finally, we obtain

$$M = M_1 + M_2 \leq h(v_M) + (n+1)M_1 \leq h(v_M) + (n+1)\frac{v_M + \mu_K}{\tau}.$$

□

Proof of Proposition 22. Let us prove that (1) implies (2) in Proposition 22. Consider $v \in \mathcal{V}$. Lemma 24 ensures that $v \in \mathbb{Z}_{d,\ell}$. Lemma 20 ensures that $v \geq -\mu_K$. It remains to prove that $v \in \mathcal{V}_{\iota(v)}$. Let M be the least positive integer such that $v \in \mathcal{V}_M$. Lemma 21 ensures that there exist $v_0 < \cdots < v_M$ in $\mathcal{V}$ such that

- $v_0 \in \mathcal{V}_0 = -\mathcal{S}(L)$;
- $v_{i+1} \in \pi(\Psi(v_i))$ for any $i \in \{0, \ldots, M-1\}$;
- $v_M = v$.

It follows from Lemma 28 that $M \leq \iota(v)$. Since $(\mathcal{V}_i)_{i\geq 0}$ is nondecreasing by Theorem 17, we have $v \in \mathcal{V}_M \subset \mathcal{V}_{\iota(v)}$. This proves that (1) implies (2) in Proposition 22. The converse implication is obvious. □

5. An algorithm for computing a positive lower bound on $\epsilon(v)$

We retain the notations from the previous section. In addition, we set $\mu_0 = -\infty$.

5.1. **Structure of the algorithm.** The aim of this section is to present an algorithm for computing a positive lower bound on $\epsilon(v)$ for any given $v \in \mathbb{Z}_{d,\ell}$. This algorithm is recursive and its structure is as follows.

- The base case corresponds to the case when v belongs to $]-\infty, -\mu_K] \cap \mathbb{Z}_{d,\ell}$. This case presents no difficulty because $\epsilon(v)$ can be computed explicitly as we will see in Proposition 29.
- The recursive step is organized as follows. We consider an element v of $]-\mu_K, +\infty[\cap \mathbb{Z}_{d,\ell}$ and we distinguish two cases:
  - if $v \notin \{-\mu_{K-1}, \ldots, -\mu_1\}$, then there exists $k \in \{1, \ldots, K\}$ such that $v \in]-\mu_k, -\mu_{k-1}[\cap \mathbb{Z}_{d,\ell}$ and we will see in Proposition 35 how to compute a positive lower bound on $\epsilon(v)$ from positive lower bounds on $\epsilon(w)$ for finitely many explicit $w \in]-\infty, -\mu_k] \cap \mathbb{Z}_{d,\ell}$;
  - if $v = -\mu_{k-1}$ for some $k \in \{2, \ldots, K\}$, then we will see in Proposition 38 how to compute a positive lower bound on $\epsilon(v) = \epsilon(-\mu_{k-1})$ from positive lower bounds on $\epsilon(w)$ for finitely many explicit $w \in]-\infty, -\mu_{k-1}[\cap \mathbb{Z}_{d,\ell}$.

The algorithm is presented in pseudo-code form in Algorithm 41 in section 5.5. The theoretical results mentioned above, namely Proposition 29, Proposition 35 and Proposition 38, on which the algorithm is based, are the subject of the next three sections.

5.2. **Theoretical result for the base case.** The following result allows us to compute a lower bound on $\epsilon(v)$ for any given $v \in]-\infty, -\mu_K] \cap \mathbb{Z}_{d,\ell}$.

Proposition 29. *We have:*

- *for any* $v \in]-\infty, -\mu_K[\cap \mathbb{Z}_{d,\ell}$, $\epsilon(v) = -\mu_K - v > 0$;
- $\epsilon(-\mu_K) = \min(\mathcal{V}_1 \setminus \{-\mu_K\}) + \mu_K$.

Therefore, a lower bound on $\epsilon(v)$ *is given by*

- $-\mu_K - v$ *if* $v \in]-\infty, -\mu_K[\cap \mathbb{Z}_{d,\ell}$;
- $\min(\mathcal{V}_1 \setminus \{-\mu_K\}) + \mu_K$ *if* $\mathcal{V}_1 \setminus \{-\mu_K\} \neq \emptyset$ *and* 1 *otherwise*[5], *if* $v = -\mu_K$.

Example 30. In Section 9, we will illustrate our algorithm for computing a positive lower bound on $\epsilon(v)$, namely Algorithm 41 presented in section 5.5 below, on the operator defined by (14). We will have to compute positive lower bounds on $\epsilon(-\frac{3}{4})$ and $\epsilon(-\frac{1}{2})$. Let us explain how this can be done using Proposition 29. We will see in section 9.1 that $K = 2$, $\mu_1 = 0$ and $\mu_2 = \frac{1}{2}$. Since $-\frac{3}{4} < -\mu_2$, Proposition 29 ensures that

$$\epsilon(-\frac{3}{4}) = -\mu_2 - (-\frac{3}{4}) = -\frac{1}{2} + \frac{3}{4} = \frac{1}{4}.$$

Moreover, we will see in section 9.3 that $\mathcal{V}_1 = \{-\frac{1}{2}, -\frac{1}{4}, 0, 1\}$, so $\mathcal{V}_1 \setminus \{-\mu_2\} = \{-\frac{1}{4}, 0, 1\} \neq \emptyset$ and Proposition 29 ensures that

$$\epsilon(-\frac{1}{2}) = \epsilon(-\mu_2) = \min(\mathcal{V}_1 \setminus \{-\mu_2\}) + \mu_2 = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}.$$

Proof of Proposition 29. Lemma 20 ensures that $\min \mathcal{V} = -\mu_K$. Therefore,

- for all $v \in]-\infty, -\mu_K[\cap \mathbb{Z}_{d,\ell}$, we have $\epsilon(v) = \min \mathcal{V}_{>v} - v = -\mu_K - v > 0$;
- $\epsilon(-\mu_K) = \min \mathcal{V}_{>-\mu_K} + \mu_K = \min(\mathcal{V} \setminus \{-\mu_K\}) + \mu_K$ and the desired equality $\epsilon(-\mu_K) = \min(\mathcal{V}_1 \setminus \{-\mu_K\}) + \mu_K$ follows from Lemma 31 below.

□

Lemma 31. *We have* $\min \mathcal{V} \setminus \{-\mu_K\} = \min \mathcal{V}_1 \setminus \{-\mu_K\}$.

Proof. We set $w = \min \mathcal{V} \setminus \{-\mu_K\}$. If $\mathcal{V} \setminus \{-\mu_K\} = \emptyset$, then $\mathcal{V}_1 \setminus \{-\mu_K\} = \emptyset$ as well and the equality $w = \min \mathcal{V}_1 \setminus \{-\mu_K\}$ holds in this case. From now on, we assume that $\mathcal{V} \setminus \{-\mu_K\} \neq \emptyset$. In order to prove the lemma, it is sufficient to prove that $w \in \mathcal{V}_1$. Let M be the least element of $\mathbb{Z}_{\geq 0}$ such that $w \in \mathcal{V}_M$. If $M = 0$, then $w \in \mathcal{V}_0 \subset \mathcal{V}_1$ and the lemma is proved in this case. Suppose now that $M \geq 1$. We want to prove that $M = 1$. According to Lemma 21, there exist $v_0, v_1, \ldots, v_M \in \mathcal{V}$ such that $v_0 \in -\mathcal{S}(L)$, $v_0 < v_1 < \cdots < v_M$ and $v_M = w$. Since $\min -\mathcal{S}(L) = -\mu_K$ and $v_0 \in -\mathcal{S}(L)$, the fact that $v_1 > v_0$ implies that $v_1 > -\mu_K$ and, hence, $v_1 \in \mathcal{V} \setminus \{-\mu_K\}$. It follows that $v_1 \geq \min \mathcal{V} \setminus \{-\mu_K\} = w = v_M$ and, hence, $M = 1$. □

[5]Here, 1 is an arbitrary choice, any positive value would work.

5.3. **Theoretical results for the recursive step: case $v \notin \{-\mu_{K-1}, \dots, -\mu_1\}$.** In this section, we consider $k \in \{1, \dots, K\}$ and we assume that we are able to compute a positive lower bound $\breve{\epsilon}(w)$ on $\epsilon(w)$ for any $w \in]-\infty, -\mu_k] \cap \mathbb{Z}_{d,\ell}$. Our aim is to explain how one can compute, for any $v \in]-\mu_k, -\mu_{k-1}[\cap \mathbb{Z}_{d,\ell}$, a positive lower bound on $\epsilon(v)$ by using finitely many of the values $\breve{\epsilon}(w)$, with $w \in]-\infty, -\mu_k] \cap \mathbb{Z}_{d,\ell}$.

Our approach, detailed in Proposition 35 below, relies on a labeled rooted tree $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ that we shall now introduce. Its definition involves the finite (possibly empty) sets defined, for any $w \in \mathbb{Z}_{d,\ell}$, by

$$\Delta(w) = \left\{ \frac{\psi(w) - \beta}{\ell^\alpha} \mid (\ell^\alpha, \beta) \in \mathcal{P}(L) \right\} \setminus \{w\},$$

In geometric terms, $\Delta(w)$ is the set of $w' \in \mathbb{Q} \setminus \{w\}$ for which there exists a point in $\mathcal{P}(L)$ whose projection along a line of slope $-w'$ onto the y-axis is the point with coordinates $(0, \psi(w))$. It will also be useful to keep in mind that, in more computational terms, $\Delta(w)$ is the set of $w' \in \mathbb{Q} \setminus \{w\}$ such that $\psi(w) \in \Psi(w')$ and that, according to the discussion at the beginning of section 3.4, $\Psi(w')$ is a natural receptacle for $\operatorname{supp} L(z^{w'})$.

Definition 32. *Let $k \in \{1, \dots, K\}$ and $v \in]-\mu_k, -\mu_{k-1}[\cap \mathbb{Z}_{d,\ell}$. The labeled rooted tree $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ alluded to above is uniquely defined by the following properties:*

- *the labels of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ belong to $\mathbb{Q}$;*
- *the label of the root of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ is v;*
- *if a vertex has label $w < -\mu_k + \breve{\epsilon}(-\mu_k)$, then it has no child, i.e., it is a leaf;*
- *if a vertex has label $w \geq -\mu_k + \breve{\epsilon}(-\mu_k)$, then it has $\sharp\Delta(w)$ children labeled by the elements of $\Delta(w)$.*

Note that:

- In the last case of Definition 32, since $w > -\mu_K$, it follows from Lemma 33 below that $\Delta(w) \neq \emptyset$ and, hence, the vertex is not a leaf.
- In the tree $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$, children have smaller labels than their parent; this is a direct consequence (of the first assertion) of Lemma 34 below.

The following two lemmas give useful properties of the set $\Delta(w)$.

Lemma 33. *If the set $\Delta(w)$ is empty for some $w \in \mathbb{Z}_{d,\ell}$, then $K = 1$ and $w = -\mu_K = -\mu_1$.*

Proof. The following properties are obviously equivalent:

- $\Delta(w)$ is empty;
- for all $(\ell^\alpha, \beta) \in \mathcal{P}(L)$, $\frac{\psi(w)-\beta}{\ell^\alpha} = w$;
- for all $(\ell^\alpha, \beta) \in \mathcal{P}(L)$, $\psi(w) = \ell^\alpha w + \beta$;
- $\Psi(w) = \{\psi(w)\}$;
- $\Psi(w)$ is a singleton.

Since $\Psi(w)$ is the set of ordinates of the projection of the elements of $\mathcal{P}(L)$ along a line of slope $-w$ onto the y-axis, the fact that $\Psi(w)$ is a singleton is equivalent to the fact that the elements of $\mathcal{P}(L)$ belong to a single line of slope $-w$. In that case, $\mathcal{N}(L)$ has a single slope and this slope is equal to $-w$, *i.e.*, $K=1$ and $w=-\mu_K=-\mu_1$. $\square$

Lemma 34. *Let $w,w' \in \mathbb{Z}_{d,\ell}$. If $w' \in \Delta(w)$, then $w' < w$ and $w \in \pi(\Psi(w'))$. Reciprocally*[6]*, if $w \in \pi(\Psi(w'))$ and $w \neq w'$, then $w' \in \Delta(w)$ and $w' < w$.*

Proof. Consider $w' \in \Delta(w)$. There exists $(\ell^\alpha, \beta) \in \mathcal{P}(L)$ such that

$$w' = \frac{\psi(w)-\beta}{\ell^\alpha}.$$

Thus, we have $\psi(w) = \ell^\alpha w' + \beta$ and, hence, since π and ψ are inverse of each other by Lemma 7, we have $w = \pi(\ell^\alpha w' + \beta) \in \pi(\Psi(w'))$. Then, it follows from Lemma 15 that $w' \leq w$. Since $w' \neq w$, we have $w' < w$.

Suppose now that $w \in \pi(\Psi(w'))$ and $w \neq w'$. Then, $w = \pi(\ell^\alpha w' + \beta)$ for some $(\ell^\alpha,\beta) \in \mathcal{P}(L)$. Since π and ψ are inverse of each other by Lemma 7, we have $\psi(w) = \ell^\alpha w' + \beta$ and, hence, $w' = \frac{\psi(w)-\beta}{\ell^\alpha}$. Since $w' \neq w$, we get $w' \in \Delta(w)$. The fact that $w' < w$ follows from the first part of the lemma. $\square$

As announced above, the following result shows how to compute, for any $v \in]-\mu_k, -\mu_{k-1}[\cap \mathbb{Z}_{d,\ell}$, a positive lower bound on $\epsilon(v)$ by using finitely many $\breve{\epsilon}(w)$ with $w \in]-\infty, -\mu_k] \cap \mathbb{Z}_{d,\ell}$. Our approach relies on the labeled rooted tree $\epsilon(\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v))$ obtained from $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ by applying ϵ to its labels. It will be used in the following way: we will explain how to compute a positive lower bound on the label of any leaf of $\epsilon(\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v))$ and how to compute a positive lower bound on the label of any vertex of $\epsilon(\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v))$ from positive lower bounds on its children; this will enable us to compute a positive lower bound on the label of any vertex of $\epsilon(\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v))$ and, in particular, on the label of its root, which is nothing but $\epsilon(v)$.

Proposition 35. *Let $k \in \{1, \ldots, K\}$ and $v \in]-\mu_k, -\mu_{k-1}[\cap \mathbb{Z}_{d,\ell}$.*

(i) The tree $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ is finite and its height is less than or equal to

$$(n+1)\frac{v+\mu_k}{\min\{\breve{\epsilon}(-\mu_k), (d\ell^n)^{-1}\}} + h(v) + 1. \tag{51}$$

(ii) Consider a leaf of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ with label w. Then a positive lower bound on $\epsilon(w)$ is given by $\breve{\epsilon}(w)$ if $w < -\mu_k$ and by $-\mu_k + \breve{\epsilon}(-\mu_k) - w$ if $-\mu_k \leq w < -\mu_k + \breve{\epsilon}(-\mu_k)$.

(iii) Consider a vertex of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ with label w which is not a leaf. If, for each $w' \in \Delta(w)$, we have a positive lower bound $m_{w'}$ on $\epsilon(w')$, then a positive lower bound on $\epsilon(w)$ is given by the minimum of

$$\{m_{w'}\ell^{d_{w,w'}-\alpha_{k-1}} \mid w' \in \Delta(w)\} \cup \{-\mu_{k-1} - w, \min(\pi(\Psi(w)) \setminus \{w\}) - w\} \tag{52}$$

where $d_{w,w'}$ is defined by (54).

Consequently:

[6]Strictly speaking, the reciprocal assertion should start with the hypothesis "$w' < w$ and $w \in \pi(\Psi(w'))$" instead of the seemingly weaker hypothesis "$w' \neq w$ and $w \in \pi(\Psi(w'))$". Actually, these hypotheses are equivalent because $\min \pi(\Psi(w')) = w'$ according to Lemma 15.

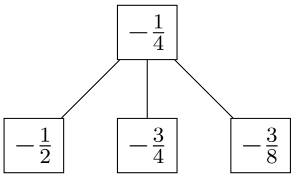

FIGURE 1

- *(i) above ensures that $\epsilon(\mathcal{T}(k,\breve{\epsilon}(-\mu_k),v))$ is a finite tree and gives an explicit upper bound on its height;*
- *(ii) above allows us to compute a positive lower bound on the label of any leaf of $\epsilon(\mathcal{T}(k,\breve{\epsilon}(-\mu_k),v))$;*
- *(iii) above allows us to compute a positive lower bound on the label of a vertex of $\epsilon(\mathcal{T}(k,\breve{\epsilon}(-\mu_k),v))$ which is not a leaf from positive lower bounds on its children.*

This allows us to compute a positive lower bound on the label of any vertex of $\epsilon(\mathcal{T}(k,\breve{\epsilon}(-\mu_k),v))$ and, in particular, a positive lower bound on the label of its root, namely $\epsilon(v)$.

Example 36. In section 9, we will have to compute a positive lower bound on $\epsilon(-\frac{1}{4})$ for the operator defined by (14). Let us explain how this can be done using Proposition 35. We will see in section 9.1 that $K=2$, $\mu_1=0$ and $\mu_2=\frac{1}{2}$. Moreover, we have seen in Example 30 that $\epsilon(-\mu_2)=\epsilon(-\frac{1}{2})=\frac{1}{4}$, thus $\breve{\epsilon}(-\mu_2)=\breve{\epsilon}(-\frac{1}{2}):=\frac{1}{4}$ is a positive lower bound on $\epsilon(-\mu_2)=\epsilon(-\frac{1}{2})$.

We have $-\frac{1}{4}\in]-\mu_2,-\mu_1[=]-\frac{1}{2},0[$, so, following the method presented in Proposition 35, in order to compute a positive lower bound on $\epsilon(-\frac{1}{4})$, we first compute the tree $\mathcal{T}(2,\breve{\epsilon}(-\mu_2),-\frac{1}{4})=\mathcal{T}(2,\frac{1}{4},-\frac{1}{4})$. The result is shown in Figure 1.

We then compute lower bounds on the labels $\epsilon(-\frac{1}{2})$, $\epsilon(-\frac{3}{4})$ and $\epsilon(-\frac{3}{8})$ of the leaves of $\epsilon(\mathcal{T}(2,\frac{1}{4},-\frac{1}{4}))$. Since $-\frac{3}{8}\geq-\mu_2=-\frac{1}{2}$, it follows from (ii) of Proposition 35 that a positive lower bound on $\epsilon(-\frac{3}{8})$ is given by

$$m_{-\frac{3}{8}}=-\mu_2+\breve{\epsilon}(-\mu_2)-(-\frac{3}{8})=-\frac{1}{2}+\frac{1}{4}+\frac{3}{8}=\frac{1}{8}.$$

Similarly, since $-\frac{1}{2}\geq-\mu_2=-\frac{1}{2}$, it follows from (ii) of Proposition 35 that a positive lower bound on $\epsilon(-\frac{1}{2})$ is given by

$$m_{-\frac{1}{2}}=-\mu_2+\breve{\epsilon}(-\mu_2)-(-\frac{1}{2})=-\frac{1}{2}+\frac{1}{4}+\frac{1}{2}=\frac{1}{4}.$$

Last, $-\frac{3}{4}<-\mu_2=-\frac{1}{2}$, and it has been shown in Example 30 that $m_{-\frac{3}{4}}=\frac{1}{4}$ is a positive lower bound on $\epsilon(-\frac{3}{4})$ respectively.

Now that we have lower bounds on the labels of the leaves of $\epsilon(\mathcal{T}(2,\frac{1}{4},-\frac{1}{4}))$, (ii) of Proposition 35 ensures that a positive lower bound on the root $\epsilon(-\frac{1}{4})$ of $\epsilon(\mathcal{T}(2,\frac{1}{4},-\frac{1}{4}))$ is given by the minimum of (52) with $w=-\frac{1}{4}$ and $k=2$. Computing this minimum requires the calculation of α_1, of $d_{-\frac{1}{4},w'}$ for $w'\in\{-\frac{1}{2},-\frac{3}{4},-\frac{3}{8}\}$ and of $\min(\pi(\Psi(-\frac{1}{4}))\setminus\{-\frac{1}{4}\})$. This presents no difficulty. Indeed, we will see in section 9.1 that $\alpha_1=1$. Moreover, using the explicit formulas for $\mathcal{P}(L)$ and π given in Section 9.1 and Section 9.2, we get

$$d_{-\frac{1}{4},-\frac{1}{2}} = \min\Big\{\alpha \in \{0,1,2\} \mid \exists(\ell^{\alpha},\beta) \in \{(1,0),(2,0),(2,1),(4,1)\},\\ -\frac{1}{2} = \frac{\psi(-\frac{1}{4})-\beta}{2^{\alpha}} = \frac{-\frac{1}{2}-\beta}{2^{\alpha}}\Big\} = 0.$$

Similar calculations show that

$$d_{-\frac{1}{4},-\frac{3}{4}} = 1, \quad d_{-\frac{1}{4},-\frac{3}{8}} = 2.$$

Last, the explicit formulas for π and Ψ given in section 9.2 show that

$$\min(\pi(\Psi(-\frac{1}{4})) \setminus \{-\frac{1}{4}\}) = -\frac{1}{8}.$$

Finally, we obtain that a positive lower bound of $\epsilon(-\frac{1}{4})$ is given by the minimum of the following numbers:

- $m_{w'}\ell^{d_{-\frac{1}{4},w'}-\alpha_1} = \frac{1}{4} \times 2^{0-1} = \frac{1}{8}$ for $w' = -\frac{1}{2}$;
- $m_{w'}\ell^{d_{-\frac{1}{4},w'}-\alpha_1} = \frac{1}{4} \times 2^{1-1} = \frac{1}{4}$ for $w' = -\frac{3}{4}$;
- $m_{w'}\ell^{d_{-\frac{1}{4},w'}-\alpha_1} = \frac{1}{8} \times 2^{2-1} = \frac{1}{4}$ for $w' = -\frac{3}{8}$;
- $0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$;
- $\min(\pi(\Psi(-\frac{1}{4})) \setminus \{-\frac{1}{4}\}) + \frac{1}{4} = -\frac{1}{8} + \frac{1}{4} = \frac{1}{8}$;

this minimum is equal to $\frac{1}{8}$. Thus, a positive lower bound of $\epsilon(-\frac{1}{4})$ is given by $\breve{\epsilon}(-\frac{1}{4}) = \frac{1}{8}$.

The proof of Proposition 35 is given below, after the following lemma.

Lemma 37. *Let $w \in \mathbb{Q}_{>-\mu_K}$ and let $k \in \{1,\ldots,K\}$ be such that $-\mu_k \leq w < -\mu_{k-1}$, with the convention $\mu_0 = -\infty$. Then, a positive lower bound on $\epsilon(w)$ is the minimum of the following set*

$$\{\epsilon(w')\ell^{d_{w,w'}-\alpha_{k-1}} \mid w' \in \Delta(w)\} \cup \{-\mu_{k-1}-w, \min(\pi(\Psi(w))\setminus\{w\})-w\} \tag{53}$$

where

$$d_{w,w'} = \min\left\{\alpha \in \{0,\ldots,n\} \mid \exists(\ell^{\alpha},\beta) \in \mathcal{P}(L), w' = \frac{\psi(w)-\beta}{\ell^{\alpha}}\right\}. \tag{54}$$

Proof. Let $w^+ = \min \mathcal{V}_{>w}$ so that $\epsilon(w) = w^+ - w$. Since $-\mu_{k-1} \in \mathcal{V}_{>w} \cup \{+\infty\}$, we have $-\mu_{k-1} \geq w^+$. Thus:

$$-\mu_k \leq w < w^+ \leq -\mu_{k-1}.$$

We shall now distinguish several cases.

Case 1: $w^+ = -\mu_{k-1}$**.** In this case, $\epsilon(w) = w^+ - w = -\mu_{k-1} - w$ and this quantity is bounded from below by the minimum of (53).

Case 2: $w^+ < -\mu_{k-1}$**.** In this case, we have $-\mu_k < w^+ < -\mu_{k-1}$ and, hence, $w^+ \notin \mathcal{V}_0 = -\mathcal{S}(L)$. It follows from Lemma 21 that there exists $w^- \in \mathcal{V}$ such that

$$w^- < w^+ \text{ and } w^+ \in \pi(\Psi(w^-)). \tag{55}$$

Since $w^+ = \min \mathcal{V}_{>w}$, the facts that $w^- \in \mathcal{V}$ and that $w^- < w^+$ ensures that $w^- \leq w$. We now distinguish two subcases.

Subcase 2.1: $w^+ < -\mu_{k-1}$ and $w^- = w$. In this case, since $w^+ \in \pi(\Psi(w^-))$ by (55), we have

$$\epsilon(w) = w^+ - w \geq \min(\pi(\Psi(w^-)) \setminus \{w^-\}) - w = \min(\pi(\Psi(w)) \setminus \{w\}) - w$$

and this quantity is bounded from below by the minimum of (53).

Subcase 2.2: $w^+ < -\mu_{k-1}$ and $w^- < w$. Since $-\mu_k \leq w < w^+ < -\mu_{k-1}$, Lemma 8 ensures that

$$w^+ = \frac{\psi(w^+) - \beta_{k-1}}{\ell^{\alpha_{k-1}}} \text{ and } w = \frac{\psi(w) - \beta_{k-1}}{\ell^{\alpha_{k-1}}}. \tag{56}$$

Furthermore, since $w^+ \in \pi(\Psi(w^-))$ by (55) and since π and ψ are inverse of each other by Lemma 7, we have $\psi(w^+) \in \Psi(w^-)$ and, hence, there exists $(\ell^\alpha, \beta) \in \mathcal{P}(L)$ such that $\psi(w^+) = \ell^\alpha w^- + \beta$. Therefore, we have

$$w^- = \frac{\psi(w^+) - \beta}{\ell^\alpha} = \frac{\ell^{\alpha_{k-1}} w^+ + \beta_{k-1} - \beta}{\ell^\alpha}. \tag{57}$$

Let

$$w' := \frac{\psi(w) - \beta}{\ell^\alpha}. \tag{58}$$

It follows from Lemma 7 that ψ is increasing so that we have

$$w' = \frac{\psi(w) - \beta}{\ell^\alpha} < \frac{\psi(w^+) - \beta}{\ell^\alpha} = w^- < w \tag{59}$$

In particular, this implies that $w' \in \Delta(w)$. Using (56), we get

$$w^+ - w = \frac{\psi(w^+) - \beta_{k-1}}{\ell^{\alpha_{k-1}}} - \frac{\psi(w) - \beta_{k-1}}{\ell^{\alpha_{k-1}}} = \frac{\psi(w^+) - \psi(w)}{\ell^{\alpha_{k-1}}}.$$

Moreover, we infer from (57) that

$$\psi(w^+) = \ell^\alpha w^- + \beta$$

and from (58) that

$$\psi(w) = \ell^\alpha w' + \beta.$$

So, we obtain

$$w^+ - w = \frac{(\ell^\alpha w^- + \beta) - (\ell^\alpha w' + \beta)}{\ell^{\alpha_{k-1}}} = \frac{\ell^\alpha (w^- - w')}{\ell^{\alpha_{k-1}}}.$$

But, since $w' < w^-$ by (59) and since $w^- \in \mathcal{V}$, we have $w^- \geq w' + \epsilon(w')$. Therefore,

$$w^+ - w = \frac{\ell^\alpha (w^- - w')}{\ell^{\alpha_{k-1}}} \geq \ell^{\alpha - \alpha_{k-1}} \epsilon(w')$$

and the latter quantity is bounded from below by the minimum of (53). □

Proof of Proposition 35. We start with the proof of (i). Since any vertex of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ has finitely many children, in order to prove (i), it is sufficient to prove that the height of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ is less than or equal to (51). Consider an arbitrary vertex of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ with label w and depth d. Let $w_0 = w, w_1, \ldots, w_d = v$ be the labels of the vertices encountered along the path from this vertex to the root of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$, so that, for any $i \in \{0, \ldots, d-1\}$, we have

$$w_i \in \Delta(w_{i+1}).$$

We claim that

$$-\mu_k + \breve{\epsilon}(-\mu_k) \leq w_1 < \cdots < w_{d-1} < w_d = v < -\mu_{k-1} \tag{60}$$

and that, for all $i \in \{0, \ldots, d-1\}$,

$$w_{i+1} \in \pi(\Psi(w_i)). \tag{61}$$

Indeed, for any $i \in \{0, \ldots, d-1\}$, we have $w_i \in \Delta(w_{i+1})$, so Lemma 34 ensures that $w_{i+1} > w_i$ and that $w_{i+1} \in \pi(\Psi(w_i))$. So, we have justified (61) and, in order to justify (60), it only remains to prove that $w_1 \geq -\mu_k + \breve{\epsilon}(-\mu_k)$. The latter inequality follows from the fact that w_1 is the label of a vertex of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$ which is not a leaf because that vertex has a vertex with label w_0 as a child.

Now, applying Lemma 27, we get

$$d - 1 < (n+1)\frac{w_d + \mu_k}{\min\{\breve{\epsilon}(-\mu_k), (d\ell^n)^{-1}\}} + h(w_d)$$

and, hence, d is less than or equal to (51). This concludes the proof of (i).

Let us now prove (ii). Since w is the label of a leaf of $\mathcal{T}(k, \breve{\epsilon}(-\mu_k), v)$, we have $w < -\mu_k + \breve{\epsilon}(-\mu_k)$. If $w < -\mu_k$, there is nothing to prove. Suppose that $-\mu_k \leq w < -\mu_k + \breve{\epsilon}(-\mu_k)$. Then $-\mu_k + \breve{\epsilon}(-\mu_k) - w > 0$. Moreover, for any $w' \in \mathcal{V}_{>w}$, we have $w' > w \geq -\mu_k$ and, hence, $w' \geq -\mu_k + \breve{\epsilon}(-\mu_k) = w + (-\mu_k + \breve{\epsilon}(-\mu_k) - w)$. We have shown that $-\mu_k + \breve{\epsilon}(-\mu_k) - w$ is a positive lower bound on $\epsilon(w)$.

Last, (iii) follows from Lemma 37. Indeed, Lemma 37 ensures that a positive lower bound on $\epsilon(v)$ is given by the minimum of

$$\{\epsilon(w')\ell^{d_{w,w'} - \alpha_{k-1}} \mid w' \in \Delta(w)\} \cup \{-\mu_{k-1} - w, \min(\pi(\Psi(w)) \setminus \{w\}) - w\}. \tag{62}$$

Since $m_{w'} \leq \epsilon(w')$ for any $w' \in \Delta(w)$, the latter minium is greater than or equal to the minimum of

$$\{m_{w'}\ell^{d_{w,w'} - \alpha_{k-1}} \mid w' \in \Delta(w)\} \cup \{-\mu_{k-1} - w, \min(\pi(\Psi(w)) \setminus \{w\}) - w\} \tag{63}$$

(we emphasize that the only difference between (62) and (63) is the first quantity, $\epsilon(w')$ versus $m_{w'}$), whence the desired result. □

5.4. **Theoretical results for the recursive step: case $v \in \{-\mu_{K-1}, \ldots, -\mu_1\}$.** In this section, we consider $k \in \{1, \ldots, K\}$ and we assume that we are able to compute a positive lower bound $\breve{\epsilon}(w)$ on $\epsilon(w)$ for any $w \in]-\infty, -\mu_{k-1}[\cap \mathbb{Z}_{d,\ell}$. The following result explains how one can compute a positive lower bound on $\epsilon(-\mu_{k-1})$ by using finitely many $\breve{\epsilon}(w)$ with $w \in]-\infty, -\mu_{k-1}[\cap \mathbb{Z}_{d,\ell}$.

Proposition 38. *Let $k \in \{2, \ldots, K\}$. A positive lower bound on $\epsilon(-\mu_{k-1})$ is given by the minimum of*

$$\begin{gathered}\{\breve{\epsilon}(w')\ell^{d_{-\mu_{k-1},w'} - \alpha_{k-2}} \mid w' \in \Delta(-\mu_{k-1})\} \\ \textit{and} \\ \{\mu_{k-1} - \mu_{k-2}, \min(\pi(\Psi(-\mu_{k-1})) \setminus \{-\mu_{k-1}\}) + \mu_{k-1}\}\end{gathered} \tag{64}$$

where $d_{-\mu_{k-1},w'}$ is defined by (54).

Remark 39. *It follows from (the first assertion of) Lemma 34 that any $w' \in \Delta(-\mu_{k-1})$ satisfies $w' < -\mu_{k-1}$, hence, it is legitimate to consider $\breve{\epsilon}(w')$ in (64).*

Example 40. In section 9, we will have to compute a positive lower bound on $\epsilon(0)$ for the operator defined by (14). Let us explain how this can be done using Proposition 38. We will see in section 9.1 that $K=2$, $\mu_1=0$ and $\mu_2=\frac{1}{2}$. Since $0=-\mu_1$, a lower bound on $\epsilon(0)$ is given by the minimum of the set (64) with $k=2$ and $-\mu_{k-1}=-\mu_1=0$. In order to compute this minimum, we have to compute $\Delta(0)$. We find $\Delta(0)=\{-\frac{1}{2},-\frac{1}{4}\}$. Then, we have to compute positive lower bounds $\breve{\epsilon}(-\frac{1}{2})$ and $\breve{\epsilon}(-\frac{1}{4})$ on $\epsilon(-\frac{1}{2})$ and $\epsilon(-\frac{1}{4})$ respectively. We have seen in Example 30 and Example 36 that we can take $\breve{\epsilon}(-\frac{1}{2})=\frac{1}{4}$ and $\breve{\epsilon}(-\frac{1}{4})=\frac{1}{8}$. Using the calculations of section 9.2, it is easily seen that the minimum of (64) with $k=2$ and $-\mu_{k-1}=-\mu_1=0$ is the minimum of the following numbers:

- $\breve{\epsilon}(w')\ell^{d_{0,w'}-\alpha_0}=\frac{1}{4}2^{1-0}=\frac{1}{2}$ for $w'=-\frac{1}{2}$;
- $\breve{\epsilon}(w')\ell^{d_{0,w'}-\alpha_0}=\frac{1}{8}2^{2-0}=\frac{1}{2}$ for $w'=-\frac{1}{4}$;
- $\mu_1-\mu_0=+\infty$;
- $\min(\pi(\Psi(-\mu_1))\setminus\{-\mu_1\})+\mu_1=\min(\pi(\Psi(0))\setminus\{0\})=1$.

This minimum is equal to $\frac{1}{2}$. Thus, a positive lower bound of $\epsilon(0)$ is given by $\breve{\epsilon}(0)=\frac{1}{2}$.

Proof. This follows immediately from Lemma 37 when replacing k with $k-1$ since $\breve{\epsilon}(w')\leq\epsilon(w')$ for any $w'\in\Delta(-\mu_{k-1})$. □

5.5. **Pseudo-code.** Here is a pseudo-code transcription of the algorithm outlined in section 5.1. Its core is the function Lower_Bound_ϵ_param defined in Algorithm 42 below.

Algorithm 41.

Input: L a Mahler operator with coefficients in $\mathbf{K}[z]$, $v\in\mathbb{Z}_{d,\ell}$.
Output: positive lower bound on $\epsilon(v)$.

---

def Lower_Bound_ϵ (L,v)
  for k **from** K **to** 1
    set $\breve{\epsilon}_k$=Lower_Bound_ϵ_param $(L,k,(\breve{\epsilon}_i)_{k+1\leq i\leq K},-\mu_k)$
  end for
  return Lower_Bound_ϵ_param $(L,0,(\breve{\epsilon}_i)_{1\leq i\leq K},v)$
end def

Algorithm 42.

Input: L a Mahler operator with coefficients in $\mathbf{K}[z]$, $K_0\in\{0,\dots,K\}$, $(\breve{\epsilon}_i)_{K_0+1\leq i\leq K}\in\mathbb{Q}_{>0}^{K-K_0}$, $v\in\mathbb{Z}_{d,\ell}$.
Output: positive lower bound on $\epsilon(v)$ provided that:

- either $K_0=K$ and $v\leq-\mu_{K_0}=-\mu_K$;
- or $K_0\in\{0,\dots,K-1\}$, $v\leq-\mu_{K_0}$ and $\breve{\epsilon}_{K_0+1},\dots,\breve{\epsilon}_K$ are positive lower bounds on $\epsilon(-\mu_{K_0+1}),\dots,\epsilon(-\mu_K)$ respectively.

---

def Lower_Bound_ϵ_param $(L,K_0,(\breve{\epsilon}_i)_{K_0+1\leq i\leq K},v)$

1 **if** $v < -\mu_K$ **then**
2 return $-\mu_K - v$
3 **end if**
4 **if** $v = -\mu_K$ **then**
5 set $S = \mathcal{V}_1 \setminus \{-\mu_K\}$
6 **if** $S \neq \emptyset$ **then**
7 return $\min S + \mu_K$
8 **otherwise**
9 return 1
10 **end if**
11 **end if**
12 **if** $v \in]-\mu_k, -\mu_{k-1}[$ for some $k \in \{K_0+1, \ldots, K\}$ **then**
13 set (w', m) =Lower_Bound_ϵ_interval $(L, k-1, (\breve{\epsilon}_i)_{k \leq i \leq K}, v)$
14 return m
15 **end if**
16 **if** $v = -\mu_{k-1}$ for some $k \in \{K_0+1, \ldots, K\}$ **then**
17 **for** $w' \in \Delta(-\mu_{k-1})$
18 set $m_{w'}$=Lower_Bound_ϵ_param $(L, k-1, (\breve{\epsilon}_i)_{k \leq i \leq K}, w')$
19 **end for**
20 return the minimum of

$$\{m_{w'} \ell^{d_{-\mu_{k-1},w'} - \alpha_{k-2}} \mid w' \in \Delta(-\mu_{k-1})\}$$

and

$$\{\mu_{k-1} - \mu_{k-2}, \min(\pi(\Psi(-\mu_{k-1})) \setminus \{-\mu_{k-1}\}) + \mu_{k-1}\}$$

21 where $d_{-\mu_{k-1},w'}$ is defined by (54).
22 **end if**
end def

In Algorithm 42:

- the lines 1–11 correspond to the base case considered in section 5.2;
- the lines 12–15 correspond to the recursive step considered in section 5.3 which can itself be encoded as a recursive algorithm, namely Algorithm 43 below;
- the lines 16–22 correspond to the recursive step considered in section 5.4.

Algorithm 43.

Input: L a Mahler operator with coefficients in $\mathbf{K}[z]$, $K_0 \in \{0, \ldots, K-1\}$, $(\breve{\epsilon}_i)_{K_0+1 \leq i \leq K} \in \mathbb{Q}_{>0}^{K-K_0}$, $w \in \mathbb{Z}_{d,\ell}$.
Output: positive lower bound on $\epsilon(w)$ provided that $w < -\mu_{K_0}$ and $\breve{\epsilon}_{K_0+1}, \ldots, \breve{\epsilon}_K$ are positive lower bounds on $\epsilon(-\mu_{K_0+1}), \ldots, \epsilon(-\mu_K)$ respectively.

def Lower_Bound_ϵ_interval $(L, K_0, (\breve{\epsilon}_i)_{K_0+1 \leq i \leq K}, w)$
if $w < -\mu_{K_0+1} + \breve{\epsilon}_{K_0+1}$ **then**
if $w < -\mu_{K_0+1}$ **then**
set m =Lower_Bound_ϵ_param $(L, K_0+1, (\breve{\epsilon}_i)_{K_0+2 \leq i \leq K}, w)$
otherwise

set $m = -\mu_{K_0+1} + \breve{\epsilon}_{K_0+1} - w$
end if
return (w, m)
end if
if $w \geq -\mu_{K_0+1} + \breve{\epsilon}_{K_0+1}$ **then**
for each $v \in \Delta(w)$
compute $\boldsymbol{b}_v$=Lower_Bound_ϵ_interval $(L, K_0, (\breve{\epsilon}_i)_{K_0+1 \leq i \leq K}, v)$
end for
set m to the minimum of

$$\{m'\ell^{d_{w,w'}-\alpha_k} \mid (w', m') = \boldsymbol{b}_v \text{ for some } v \in \Delta(w)\} \tag{65}$$
$$\cup \quad \{-\mu_k - w, \min(\pi(\Psi(w)) \setminus \{w\}) - w\}$$

where $d_{w,w'}$ is defined by (54)
return (w, m)
end if
end def

5.6. **An algorithm for computing a positive lower bound on τ.** To compute a lower bound on $\tau = \min\left\{\epsilon(-\mu_1), \ldots, \epsilon(-\mu_K), (d\ell^n)^{-1}\right\}$, we can simply run Algorithm 41 K times to compute positive lower bounds on $\epsilon(-\mu_1), \ldots, \epsilon(-\mu_K)$. However, the calculations would involve numerous redundancies. From an algorithmic point of view, it is better to use the following algorithm which eliminates these redundancies and which is an obvious modification of Algorithm 41.

Algorithm 44.

Input: L a Mahler operator with coefficients in $\mathbf{K}[z]$
Output: positive lower bound on $\tau = \min\left\{\epsilon(-\mu_1), \ldots, \epsilon(-\mu_K), (d\ell^n)^{-1}\right\}$.

---

def Lower_Bound_τ (L)
for k **from** K **to** 1
set $\breve{\epsilon}_k$= Lower_Bound_ϵ_param $(L, k, (\breve{\epsilon}_i)_{k+1 \leq i \leq K}, -\mu_k)$
end for
return the minimum of $\breve{\epsilon}_1, \ldots, \breve{\epsilon}_K$ and $\frac{1}{d\ell^n}$
end def

---

6. The property $\star_{\mathcal{V}}$

In this section, we consider a Mahler operator

$$L = a_n\phi_\ell^n + a_{n-1}\phi_\ell^{n-1} + \cdots + a_0 \tag{66}$$

with coefficients $a_0, \ldots, a_n \in \mathbf{K}[z]$ such that $a_0 a_n \neq 0$. We recall the following notation:

$$\mathrm{Sol}(L, \mathscr{H}) = \{f \in \mathscr{H} \mid L(f) = 0\}.$$

Hypothesis A. *Throughout this section, we let $\mathcal{V}$ be a subset of $\mathbb{Q}$ satisfying the following properties:*

(1) $\mathrm{Sol}(L, \mathscr{H}) \subset \mathscr{H}_{|\mathcal{V}}$*;*
(2) $\mathcal{V}$ *is well-ordered;*
(3) $-\mathcal{S}(L) \subset \mathcal{V}$*;*
(4) $\bigcup_{v \in \mathcal{V}} \pi(\Psi(v)) = \mathcal{V}$*.*

Theorem 17 ensures that such a set $\mathcal{V}$ exists.

Definition 45. *We say that a subset* $\mathcal{R}$ *of* $\mathbb{Q}$ *satisfies property* $\star_{\mathcal{V}}$ *if :*

a. $-\mathcal{S}(L) \subset \mathcal{R} \subset \mathcal{V}$*;*
b. $\bigcup_{v \in \mathcal{V} \setminus \mathcal{R}} \pi(\Psi(v)) = \mathcal{V} \setminus \mathcal{R}$*.*

The interest of property $\star_{\mathcal{V}}$ lies in the following result.

Theorem 46. *If* $\mathcal{R}$ *satisfies* $\star_{\mathcal{V}}$*, then the* $\mathbf{K}$*-linear map*

$$\bullet_{|\mathcal{R}} : \begin{array}{ccc} \mathscr{H} & \rightarrow & \mathscr{H}_{|\mathcal{R}} \\ f = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma & \mapsto & f_{|\mathcal{R}} = \sum_{\gamma \in \mathcal{R}} f_\gamma z^\gamma \end{array} \tag{67}$$

induces a $\mathbf{K}$*-linear isomorphism*

$$\mathrm{Sol}(L, \mathscr{H}) \xrightarrow{\sim} \mathcal{C}_{\mathcal{R}} \tag{68}$$

where

$$\begin{aligned} \mathcal{C}_{\mathcal{R}} &= \{ f \in \mathscr{H}_{|\mathcal{R}} \mid \pi(\operatorname{supp} L(f)) \cap \mathcal{R} = \emptyset \} \\ &= \{ f \in \mathscr{H}_{|\mathcal{R}} \mid \operatorname{supp}(L(f)) \cap \psi(\mathcal{R}) = \emptyset \}. \end{aligned}$$

The proof of this result is given in section 6.2 below. It relies on certain preliminary results gathered in the next section.

Remark 47. *In Theorem 56, we will prove that there exists a finite subset* $\mathcal{R}$ *of* $\mathbb{Q}$ *satisfying property* $\star_{\mathcal{V}}$ *and we will give an algorithm to compute it. However, in this section,* $\mathcal{R}$ *is not required to be finite.*

6.1. **Preliminary results.** Through this section, we consider a subset $\mathcal{R}$ of $\mathbb{Q}$ satisfying $\star_{\mathcal{V}}$. We introduce the following sets:

$$\Gamma = \mathcal{V} \setminus \mathcal{R} \text{ and } \Lambda = \psi(\Gamma).$$

The principal aim of this section is to prove the following result.

Proposition 48. *For all* $g \in \mathscr{H}_{|\Lambda}$*, there exists* $f \in \mathscr{H}_{|\Gamma}$ *such that* $L(f) = g$*.*

The proof of this result is given at the end of this subsection. The following lemmas are technical results used in the proof of Proposition 48. On first reading, the reader can admit these lemmas and read the proof of Proposition 48 directly.

Lemma 49. *We have:*

- $\Gamma = \pi(\Lambda)$*;*
- $\Lambda = \bigcup_{\gamma \in \Gamma} \Psi(\gamma)$*;*
- $\Gamma \cap -\mathcal{S}(L) = \emptyset$*.*

Proof. The equality $\Gamma = \pi(\Lambda)$ follows immediately from the fact that ψ and π are inverse of each other by Lemma 7.

By definition of Γ and condition b. of Definition 45, we have

$$\Gamma = \mathcal{V} \setminus \mathcal{R} = \bigcup_{v \in \mathcal{V} \setminus \mathcal{R}} \pi(\Psi(v)) = \bigcup_{v \in \Gamma} \pi(\Psi(v)).$$

Applying ψ to the latter equality and using the fact that ψ and π are inverse of each other by Lemma 7, we obtain $\Lambda = \bigcup_{\gamma\in\Gamma} \Psi(\gamma)$.

Last, we have $\Gamma \cap -\mathcal{S}(L) = \emptyset$ because $-\mathcal{S}(L) \subset \mathcal{R}$ and $\Gamma \cap \mathcal{R} = \emptyset$ by definition of Γ. □

Lemma 50. *The subsets Γ and Λ of $\mathbb{Q}$ are well-ordered.*

Proof. Since Γ is a subset of $\mathcal{V}$ which is well-ordered, Γ is well-ordered. Lemma 7 ensures that $\psi : \mathbb{Q} \to \mathbb{Q}$ is increasing. Thus, the fact that $\Lambda = \psi(\Gamma)$ is well-ordered follows from the fact that Γ is well-ordered. □

Lemma 51. *For any $g \in \mathscr{H}_{|\Lambda} \setminus \{0\}$, there exists $a \in \mathbf{K}$ such that*

$$\mathrm{val}(L(az^\gamma) - g) > \mathrm{val}\, g,$$

where $\gamma = \pi(\mathrm{val}\, g)$. In particular, $\gamma \in \pi(\Lambda) = \Gamma$.

Proof. Since $\mathrm{supp}\, g \subset \Lambda$, we have $\mathrm{val}\, g \in \Lambda$ and, hence, $\gamma = \pi(\mathrm{val}\, g) \in \pi(\Lambda)$. But, $\pi(\Lambda) = \Gamma$ and $\Gamma \cap -\mathcal{S}(L) = \emptyset$ by Lemma 49. So, $\gamma \notin -\mathcal{S}(L)$ and Lemma 11 ensures that $\mathrm{val}\, L(z^\gamma) = \psi(\gamma)$. Since ψ and π are inverse of each other by Lemma 7, we get $\mathrm{val}\, L(z^\gamma) = \mathrm{val}\, g$, *i.e.*, there exists $c \in \mathbf{K}^\times$ such that

$$L(z^\gamma) = cz^{\mathrm{val}\, g} + \text{Hahn series of higher-order valuation.}$$

Therefore, $a = c^{-1} g_{\mathrm{val}\, g}$ has the expected property. □

Lemma 52. *If $\gamma \in \Gamma$ and $\lambda \in \Lambda$ are such that, for all $x \in \Gamma \cap]-\infty, \gamma[$, $\lambda > \psi(x)$, then $\lambda \geq \psi(\gamma)$.*

Proof. Since π and ψ are inverse of each other by Lemma 7, we have $\lambda = \psi(y)$ with $y = \pi(\lambda) \in \pi(\Lambda) = \Gamma$. By hypothesis, for all $x \in \Gamma \cap]-\infty, \gamma[$, we have $\psi(y) = \lambda > \psi(x)$. Since ψ is increasing by Lemma 7, we get, for all $x \in \Gamma \cap]-\infty, \gamma[$, $y > x$. So, $y \in \Gamma \cap [\gamma, +\infty[$. In particular, we have $y \geq \gamma$ and, since ψ is increasing by Lemma 7, $\lambda = \psi(y) \geq \psi(\gamma)$ as claimed. □

Lemma 53. *Suppose that $f, f' \in \mathscr{H}_{|\Gamma}$ are such that $\mathrm{val}(L(f) - g) > \psi(y)$ and $\mathrm{val}(L(f') - g) > \psi(y')$ for some $g \in \mathscr{H}$ and some $y, y' \in \Gamma$. Then, $f = f'$ on $\Gamma \cap]-\infty, \min\{y, y'\}]$.*

Proof. We argue by contradiction. Assume on the contrary that $f \neq f'$ on $\Gamma \cap]-\infty, \min\{y, y'\}]$. Up to interchanging the roles of f and f', we can assume that $y \leq y'$ and, hence, that $f \neq f'$ on $\Gamma \cap]-\infty, y]$. Using the fact that Γ is well ordered, we can assume that y is minimal with respect to the property "$f \neq f'$ on $\Gamma \cap]-\infty, y]$". On the one hand, one can characterize y in terms of $h = f' - f \in \mathscr{H}_{|\Gamma} \setminus \{0\}$ as the minimal element of Γ such that $h \neq 0$ on $\Gamma \cap]-\infty, y]$. So, $y = \mathrm{val}\, h$. Since $\mathrm{val}\, h = y \in \Gamma$ and $\Gamma \cap -\mathcal{S}(L) = \emptyset$ by Lemma 49, we have $\mathrm{val}\, h \notin -\mathcal{S}(L)$ and it follows from Lemma 11 that

$$\mathrm{val}\, L(h) = \psi(y). \tag{69}$$

On the other hand, we have

$$L(h) = (L(f') - g) - (L(f) - g).$$

Applying the z-adic valuation val to the latter equality and using (9), we get

$$\mathrm{val}\, L(h) \geq \min\{\mathrm{val}(L(f) - g), \mathrm{val}(L(f') - g)\}.$$

But, by hypothesis, we have $\mathrm{val}(L(f)-g) > \psi(y)$ and $\mathrm{val}(L(f')-g) > \psi(y') \geq \psi(y)$, the latter inequality $\psi(y') \geq \psi(y)$ following from the facts that $y' \geq y$ and that ψ is increasing by Lemma 7. Therefore, $\mathrm{val}\, L(h) > \psi(y)$. This contradicts (69). □

Lemma 54. *Consider $y \in \Gamma \cup \{+\infty\}$ and a family $(f_x)_{x \in \Gamma \cap]-\infty, y[}$ of Hahn series such that, for all $x \in \Gamma \cap]-\infty, y[$, $f_x \in \mathscr{H}_{|\Gamma}$. Suppose that, for all $x, x' \in \Gamma \cap]-\infty, y[$, we have $f_x = f_{x'}$ on $\Gamma \cap]-\infty, \min\{x, x'\}]$. Then, there exists $f \in \mathscr{H}_{|\Gamma \cap]-\infty, y[}$ such that, for all $x \in \Gamma \cap]-\infty, y[$, $f = f_x$ on $\Gamma \cap]-\infty, x]$.*

Proof. We set, for all $x \in \Gamma \cap]-\infty, y[$, $f_x = \sum_{\gamma \in \Gamma} a_{x,\gamma} z^{\gamma}$. Set

$$f = \sum_{\gamma \in \Gamma \cap]-\infty, y[} a_{\gamma,\gamma} z^{\gamma} \in \mathscr{H}_{|\Gamma \cap]-\infty, y[}\,.$$

Let $x \in \Gamma \cap]-\infty, y[$ and $\gamma \in \Gamma \cap]-\infty, x]$. Using the hypothesis of the lemma with $x' = \gamma$, we have $f_x = f_\gamma$ on $\Gamma \cap]-\infty, \gamma]$. In particular, looking at the coefficients of z^{γ}, we obtain $a_{x,\gamma} = a_{\gamma,\gamma}$. Thus, $f = f_x$ on $\Gamma \cap]-\infty, x]$.
□

Proof of Proposition 48. Consider $g \in \mathscr{H}_{|\Lambda}$. We have to prove that there exists $\overline{f} \in \mathscr{H}_{|\Gamma}$ such that $L(\overline{f}) = g$. We split the proof in two main steps.

Step 1. Let us first prove that, for all $y \in \Gamma$, there exists $f_y \in \mathscr{H}_{|\Gamma}$ such that $\mathrm{val}(L(f_y)-g) > \psi(y)$. We argue by contradiction: we assume that this is not true, *i.e.*, that the set

$$Y = \{y \in \Gamma \mid \forall f \in \mathscr{H}_{|\Gamma}, \mathrm{val}(L(f)-g) \leq \psi(y)\}$$

is nonempty. Since Y is a nonempty subset of the well-ordered set Γ, it has a minimal element $y_{\min}$.

We claim that there exists $f \in \mathscr{H}_{|\Gamma \cap]-\infty, y_{\min}[}$ such that

$$\mathrm{val}(L(f)-g) > \psi(x) \tag{70}$$

for all $x \in \Gamma \cap]-\infty, y_{\min}[$, and such that

$$\mathrm{val}(L(f)-g) \in \Lambda. \tag{71}$$

Indeed, for all $x \in \Gamma \cap]-\infty, y_{\min}[$, we have $x \in \Gamma \setminus Y$ and, hence, there exists $f_x \in \mathscr{H}_{|\Gamma}$ such that

$$\mathrm{val}(L(f_x)-g) > \psi(x). \tag{72}$$

According to Lemma 53, we have, for any $x, x' \in \Gamma \cap]-\infty, y_{\min}[$, $f_x = f_{x'}$ on $\Gamma \cap]-\infty, \min\{x, x'\}]$. Lemma 54 ensures that there exists $f \in \mathscr{H}_{|\Gamma \cap]-\infty, y_{\min}[}$ such that, for all $x \in \Gamma \cap]-\infty, y_{\min}[$, $f = f_x$ on $\Gamma \cap]-\infty, x]$. Let us prove that f satisfies (70) and (71). Let us first note that, for all $x \in \Gamma \cap]-\infty, y_{\min}[$,

$$\mathrm{val}\, L(f-f_x) \geq \psi(\mathrm{val}(f-f_x)) > \psi(x); \tag{73}$$

indeed, the first inequality follows from Lemma 11, the second inequality follows from the facts that $\mathrm{val}(f-f_x) > x$ and that ψ is increasing by

Lemma 7. Using (9) and, then, the inequalities (72) and (73), we get, for all $x \in \Gamma \cap]-\infty, y_{\min}[$,

$$\begin{aligned}(74)\quad \operatorname{val}(L(f)-g) &= \operatorname{val}(L(f-f_x)+L(f_x)-g)\\ &\geq \min\{\operatorname{val} L(f-f_x), \operatorname{val}(L(f_x)-g)\} > \psi(x).\end{aligned}$$

This justifies (70). Moreover, we have $L(f)-g \neq 0$ because Y is nonempty, so

$$\operatorname{val}(L(f)-g) \in \operatorname{supp}(L(f)-g) \subset \operatorname{supp} L(f) \cup \operatorname{supp} g \subset \bigcup_{\gamma \in \Gamma} \Psi(\gamma) \cup \Lambda \subset \Lambda,$$

the latter two inclusions following from Lemma 14 and Lemma 49 respectively. This justifies (71) and, hence, our claim toward the existence of f.

We fix $f \in \mathscr{H}_{|\Gamma \cap]-\infty, y_{\min}[}$ satisfying (70) and (71). We can apply Lemma 52 to $\lambda = \operatorname{val}(L(f)-g)$ and to $\gamma = y_{\min}$ and we obtain

$$\operatorname{val}(L(f)-g) \geq \psi(y_{\min}).$$

We have already seen that $L(f)-g \neq 0$ and that $\operatorname{supp}(L(f)-g) \subset \Lambda$. So, Lemma 51 ensures that there exists $a \in \mathbf{K}$ and $\gamma \in \Gamma$ such that

$$\operatorname{val}(L(az^\gamma)+L(f)-g) > \operatorname{val}(L(f)-g) \geq \psi(y_{\min}).$$

Therefore, $f' = az^\gamma + f \in \mathscr{H}_{|\Gamma}$ satisfies

$$\operatorname{val}(L(f')-g) > \psi(y_{\min}).$$

This contradicts the fact that $y_{\min}$ belongs to Y. So, Y is empty and, hence, we have proved that, for all $y \in \Gamma$, there exists $f_y \in \mathscr{H}_\Gamma$ such that $\operatorname{val}(L(f_y)-g) > \psi(y)$.

Step 2. Lemma 53 ensures that, for all $y, y' \in \Gamma$, we have $f_{y'} = f_y$ on $\Gamma \cap]-\infty, \min\{y, y'\}]$. According to Lemma 54 applied with $y = +\infty$, there exists $\overline{f} \in \mathscr{H}_\Gamma$ such that, for all $y \in \Gamma$, $\overline{f} = f_y$ on $\Gamma \cap]-\infty, y]$. Arguing as we did above for proving (70), we see that, for all $y \in \Gamma$, $\operatorname{val}(L(\overline{f})-g) > \psi(y)$. This implies that $L(\overline{f})-g = 0$ because, otherwise, $\operatorname{val}(L(\overline{f})-g)$ would belong to $\operatorname{supp}(L(\overline{f})-g) \subset \Lambda$ but not to $\psi(\Gamma)$ and this would contradict the fact that $\Lambda = \psi(\Gamma)$ by definition. $\square$

6.2. **Proof of Theorem 46.** We recall the following notations introduced in section 6.1:

$$\Gamma = \mathcal{V} \setminus \mathcal{R}, \quad \Lambda = \psi(\Gamma)$$

and

$$\mathcal{C}_\mathcal{R} = \{f \in \mathscr{H}_{|\mathcal{R}} \mid \operatorname{supp}(L(f)) \cap \psi(\mathcal{R}) = \emptyset\}.$$

Proving Theorem 46 is equivalent to proving the following properties relative to the $\mathbf{K}$-linear map $\bullet_{|\mathcal{R}}$ defined by (67): $\bullet_{|\mathcal{R}}(\operatorname{Sol}(L, \mathscr{H})) = \mathcal{C}_\mathcal{R}$ and $\ker(\bullet_{|\mathcal{R}}) \cap \operatorname{Sol}(L, \mathscr{H}) = \{0\}$. Before proving these properties, note that

$$(75)\quad \mathcal{C}_\mathcal{R} = \{f \in \mathscr{H}_{|\mathcal{R}} \mid \operatorname{supp} L(f) \subset \Lambda\} = \{f \in \mathscr{H}_{|\mathcal{R}} \mid L(f) \in \mathscr{H}_{|\Lambda}\}.$$

Indeed, since $\Lambda = \psi(\Gamma) = \psi(\mathcal{V} \setminus \mathcal{R}) = \psi(\mathcal{V}) \setminus \psi(\mathcal{R})$, in order to prove the equality (75), it is sufficient to prove that, for any $f \in \mathscr{H}_{|\mathcal{R}}$, we have $\operatorname{supp} L(f) \subset \psi(\mathcal{V})$. As a matter of fact, the latter property is true since $\operatorname{supp} L(f) \subset \bigcup_{v \in \operatorname{supp} f} \Psi(v)$ by Lemma 14, $\bigcup_{v \in \operatorname{supp} f} \Psi(v) \subset \bigcup_{v \in \mathcal{V}} \Psi(v)$ because $\operatorname{supp} f \subset \mathcal{R} \subset \mathcal{V}$ and $\bigcup_{v \in \mathcal{V}} \Psi(v) = \psi(\mathcal{V})$ by (4) of Hypothesis A.

Proof of $\bullet_{|\mathcal{R}}(\mathrm{Sol}(L,\mathscr{H})) \subset \mathcal{C}_{\mathcal{R}}$. Consider $f \in \mathrm{Sol}(L,\mathscr{H})$. Consider the decomposition $f = f_{|\mathcal{R}} + f_{|\gamma}$. Applying L to this equality, we get $0 = L(f_{|\mathcal{R}}) + L(f_{|\gamma})$, so

$$L(f_{|\mathcal{R}}) = -L(f_{|\gamma}). \tag{76}$$

It follows from Lemma 14 that $\operatorname{supp} L(f_{|\gamma}) \subset \bigcup_{v\in\gamma} \Psi(v) = \Lambda$, the latter equality coming from Lemma 49. Therefore,

$$L(f_{|\mathcal{R}}) = -L(f_{|\gamma}) \in \mathscr{H}_{|\Lambda}.$$

Using (75), this proves that the image of f by $\bullet_{|\mathcal{R}}$ belongs to $\mathcal{C}_{\mathcal{R}}$.

Proof of $\mathcal{C}_{\mathcal{R}} \subset \bullet_{|\mathcal{R}}(\mathrm{Sol}(L,\mathscr{H}))$. Consider $f_0 \in \mathcal{C}_{\mathcal{R}}$. It follows from (75) that $L(f_0) \in \mathscr{H}_{|\Lambda}$. Proposition 48 ensures that there exists $f_1 \in \mathscr{H}_{|\Gamma}$ such that $L(f_1) = -L(f_0)$. Then $f = f_0 + f_1$ belongs to $\mathrm{Sol}(L,\mathscr{H})$ and its image by $\bullet_{|\mathcal{R}}$ is f_0.

Proof of $\ker(\bullet_{|\mathcal{R}}) \cap \mathrm{Sol}(L,\mathscr{H}) = \{0\}$. Let $f \in \ker(\bullet_{|\mathcal{R}}) \cap \mathrm{Sol}(L,\mathscr{H})$. Then, f belongs to $\mathscr{H}_{|\Gamma}$ and satisfies $L(f) = 0$. Lemma 12 ensures that $\operatorname{val} f \in -\mathcal{S}(L) \cup \{+\infty\}$. But, $\operatorname{val} f \in \Gamma \cup \{+\infty\}$ and $-\mathcal{S}(L) \cap \Gamma = \emptyset$ by Lemma 49. So, $\operatorname{val} f = +\infty$ and, hence, $f = 0$.

This concludes the proof of Theorem 46.

7. Computing an $\mathcal{R}$ containing $\mathcal{E}$ and satisfying $\star_{\mathcal{V}}$.

We use the notations of section 6: we consider the operator L given by (8) and we let $\mathcal{V}$ be a subset of $\mathbb{Q}$ satisfying Hypothesis A. Moreover, we let $\mathcal{E}$ be a finite subset of $\mathcal{V}$.

Definition 55. *We say that a set $\mathcal{R}$ satisfies property $\star_{\mathcal{E},\mathcal{V}}$ if $\mathcal{E} \subset \mathcal{R}$ and $\mathcal{R}$ satisfies property $\star_{\mathcal{V}}$.*

We shall now give a recursive construction of a finite set satisfying $\star_{\mathcal{E},\mathcal{V}}$.

Theorem 56. *The sequence $(\mathcal{R}_i)_{i\geq 0}$ of subsets of $\mathcal{V}$ recursively defined by*

$$\mathcal{R}_0 = \mathcal{E} \cup -\mathcal{S}(L)$$

and, for all $i \geq 0$,

$$\mathcal{R}_{i+1} = \{v \in \mathcal{V} \mid \pi(\Psi(v)) \cap \mathcal{R}_i \neq \emptyset\} \tag{77}$$

is an eventually constant nondecreasing sequence of finite sets. The above recursive definition formula can be rewritten as follows, for all $i \geq 0$:

$$\mathcal{R}_{i+1} = \bigcup_{(\ell^{\alpha},\beta)\in\mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}. \tag{78}$$

Moreover, $\mathcal{R} = \bigcup_{i\geq 0} \mathcal{R}_i$ is a finite set which satisfies $\star_{\mathcal{E},\mathcal{V}}$.

Example 57. The sets $\mathcal{R}_i$ and $\mathcal{R}$ are computed in section 9.5 for the operator L given by (14).

Proof.
Proof of the equality (78). This equality follows from the following chain of equalities:

$$\begin{aligned}\{v \in \mathcal{V} \mid \pi(\Psi(v)) \cap \mathcal{R}_i \neq \emptyset\} &= \{v \in \mathcal{V} \mid \Psi(v) \cap \psi(\mathcal{R}_i) \neq \emptyset\} \\ &= \{v \in \mathcal{V} \mid \exists (\ell^\alpha, \beta) \in \mathcal{P}(L), v\ell^\alpha + \beta \in \psi(\mathcal{R}_i)\} \\ &= \bigcup_{(\ell^\alpha,\beta)\in\mathcal{P}(L)} \{v \in \mathcal{V} \mid v\ell^\alpha + \beta \in \psi(\mathcal{R}_i)\} \\ &= \bigcup_{(\ell^\alpha,\beta)\in\mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}.\end{aligned}$$

Proof of the fact that the $\mathcal{R}_i$ are finite sets. We argue by induction on $i \geq 0$. The base case $i = 0$ is clear. We now proceed with the inductive step. Let us assume that $\mathcal{R}_i$ is finite for some $i \geq 0$. Then, $\psi(\mathcal{R}_i)$ is finite, so, for all $(\ell^\alpha, \beta) \in \mathcal{P}(L)$, $\ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}$ is finite. Since $\mathcal{P}(L)$ is finite, the equation (78) shows that $\mathcal{R}_{i+1}$ is finite as well. This concludes the proof.

Proof of the fact that the sequence $(\mathcal{R}_i)_{i\geq 0}$ is nondecreasing. Consider $i \in \mathbb{Z}_{\geq 0}$. Lemma 15 ensures that, for all $v \in \mathcal{R}_i$, we have $v \in \pi(\Psi(v))$, so $v \in \pi(\Psi(v)) \cap \mathcal{R}_i$ and, hence, $v \in \mathcal{R}_{i+1}$. This shows that $\mathcal{R}_i \subset \mathcal{R}_{i+1}$.

Proof of the fact that $(\mathcal{R}_i)_{i\geq 0}$ is eventually constant. We argue by contradiction: we assume that $(\mathcal{R}_i)_{i\geq 0}$ is not eventually constant. Then, $\mathcal{R} = \bigcup_{i\geq 0} \mathcal{R}_i$ is infinite because $(\mathcal{R}_i)_{i\geq 0}$ is nondecreasing. For any $v \in \mathcal{V}$, we consider the sequence $(\mathcal{R}_i(v))_{i\geq 0}$ of subsets of $\mathcal{V}$ defined by

- $\mathcal{R}_0(v) = \{v\}$;
- $\forall i \geq 0$, $\mathcal{R}_{i+1}(v) = \{w \in \mathcal{V} \mid \pi(\Psi(w)) \cap \mathcal{R}_i(v) \neq \emptyset\}$.

Then, $(\mathcal{R}_i(v))_{i\geq 0}$ is a nondecreasing sequence of finite sets (for the same reasons that $(\mathcal{R}_i)_{i\geq 0}$ is an nondecreasing sequence of finite sets). We set $\mathcal{R}(v) = \bigcup_{i\geq 0} \mathcal{R}_i(v)$. Note that :

- we have $\mathcal{R} = \bigcup_{v\in\mathcal{R}_0} \mathcal{R}(v)$; since $\mathcal{R}$ is infinite and $\mathcal{R}_0$ is finite, there exists $v_0 \in \mathcal{R}_0$ such that $\mathcal{R}(v_0)$ is infinite;
- we have $\mathcal{R}(v_0) = \{v_0\} \cup \bigcup_{v\in\mathcal{R}_1(v_0)\setminus\{v_0\}} \mathcal{R}(v)$; since $\mathcal{R}(v_0)$ is infinite and $\mathcal{R}_1(v_0)$ is finite, there exists $v_1 \in \mathcal{R}_1(v_0) \setminus \{v_0\}$ such that $\mathcal{R}(v_1)$ is infinite;
- we have $\mathcal{R}(v_1) = \{v_1\} \cup \bigcup_{v\in\mathcal{R}_1(v_1)\setminus\{v_1\}} \mathcal{R}(v)$; since $\mathcal{R}(v_1)$ is infinite and $\mathcal{R}_1(v_1)$ is finite, there exists $v_2 \in \mathcal{R}_1(v_1) \setminus \{v_1\}$ such that $\mathcal{R}(v_2)$ is infinite.

Iterating this construction, we see that there exists a sequence $(v_i)_{i\geq 0}$ of elements of $\mathcal{V}$ such that, for all $i \geq 0$, $v_{i+1} \in \mathcal{R}_1(v_i) \setminus \{v_i\}$. Therefore, we have $v_i \in \pi(\Psi(v_{i+1}))$ and $v_{i+1} \neq v_i$ so $v_{i+1} < v_i$ by Lemma 34 applied to $w = v_i$ and $w' = v_{i+1}$ (we draw the reader's attention to the fact that $v_i \in \pi(\Psi(v_{i+1}))$ and not the opposite, as it was in section 4). So, the sequence $(v_i)_{i\geq 0}$ is decreasing. This contradicts the fact that $\mathcal{V}$ is well-ordered. Thus, the sequence $(\mathcal{R}_i)_{i\in\mathbb{Z}_{\geq 0}}$ is eventually constant. In particular, $\mathcal{R} = \mathcal{R}_{i_0}$ for some $i_0 \geq 0$ and, hence, $\mathcal{R}$ is a finite set.

Proof of the fact that $\mathcal{R}$ satisfies property $\star_{\mathcal{V}}$. The fact that $\mathcal{R}$ satisfies property a. of Definition 45 is obvious. Moreover, if $v \in \mathcal{V} \setminus \mathcal{R}$, then, for all

$i \geq 0$, we have $\pi(\Psi(v)) \cap \mathcal{R}_i = \emptyset$ and, hence, $\pi(\Psi(v)) \cap \mathcal{R} = \emptyset$. Thus,

$$\bigcup_{v \in \mathcal{V} \setminus \mathcal{R}} \pi(\Psi(v)) \cap \mathcal{R} = \emptyset. \tag{79}$$

But,

$$\mathcal{V} \setminus \mathcal{R} \subset \bigcup_{v \in \mathcal{V} \setminus \mathcal{R}} \pi(\Psi(v)) \subset \mathcal{V}; \tag{80}$$

indeed, the first inclusion follows from the first assertion of Lemma 15 and the second inclusion follows from property (4) of Hypothesis A. Combining (79) and (80), we get

$$\bigcup_{v \in \mathcal{V} \setminus \mathcal{R}} \pi(\Psi(v)) = \mathcal{V} \setminus \mathcal{R}.$$

So, $\mathcal{R}$ satisfies property b. of Definition 45. □

Let $d \in \mathbb{Z}_{\geq 1}$ be a common denominator of the slopes of L and let $\mathcal{V}$ be the set given by Theorem 17. The following result gives an upper bound on the least $i \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{R}_i = \mathcal{R}$ when $\mathcal{V}$ is the set given by Theorem 17.

Proposition 58. *We let $\mathcal{V}$ be the set given by Theorem 17, we let $\mathcal{E} \subset \mathcal{V}$ be a finite set and we let $(\mathcal{R}_i)_{i \geq 0}$ and $\mathcal{R}$ be the sets given by Theorem 56. We let $H \in \mathbb{Z}_{\geq 0}$ be such that*[7] *$d\ell^H \mathcal{E} \subset \mathbb{Z}$ and $N \in \mathbb{Q}_{\geq 0}$ be such that $\mathcal{E} \cup -\mathcal{S}(L) \subset \mathbb{Q}_{\leq N}$. Let τ be the number defined by (38), namely*

$$\tau = \min\left\{\epsilon(-\mu_1), \ldots, \epsilon(-\mu_K), (d\ell^n)^{-1}\right\} \in \mathbb{Q}_{>0}.$$

We set

$$c = \left\lfloor (n+1)\frac{N + \mu_K}{\tau} \right\rfloor + H.$$

Then, for any $i \in \mathbb{Z}_{\geq c}$, we have $\mathcal{R}_i = \mathcal{R}$.

Proof. Theorem 56 guarantees that the sequence $(\mathcal{R}_i)_{i \geq 0}$ is nondecreasing and eventually constant. Let M be the least element of $\mathbb{Z}_{\geq 0}$ such that $\mathcal{R}_M = \mathcal{R}_{M+1}$. It follows easily from the definition of $(\mathcal{R}_i)_{i \geq 0}$ that, for all $i \in \mathbb{Z}_{\geq M}$, $\mathcal{R}_i = \mathcal{R}$. In order to conclude the proof, it is thus sufficient to prove that $c \geq M$. Let us prove this. We have $c \geq 0$, so the result holds if $M = 0$. Otherwise, assume $M \geq 1$. The set $\mathcal{R}_M \setminus \mathcal{R}_{M-1}$ being nonempty, one can consider $v_0 \in \mathcal{R}_M \setminus \mathcal{R}_{M-1}$. It follows from the definition of $(\mathcal{R}_i)_{i \geq 0}$, that there exist $v_1, \ldots, v_M \in \mathcal{V}$ such that, for all $i \in \{0, \ldots, M-1\}$,

$$v_{i+1} \in \pi(\Psi(v_i)) \cap \mathcal{R}_{M-1-i}.$$

We claim that, for any $i \in \{0, \ldots, M-1\}$, we have $v_i \notin \mathcal{R}_{M-1-i}$. Indeed, assume on the contrary that there exists $i \in \{0, \ldots, M-1\}$ such that $v_i \in \mathcal{R}_{M-1-i}$. Without loss of generality, we can assume that i is the least element of $\{0, \ldots, M-1\}$ satisfying the latter property. Our choice of v_0 guaranties that $i \neq 0$. We have $v_i \in \mathcal{R}_{M-1-i}$ and, by construction, $v_i \in \pi(\Psi(v_{i-1})) \cap \mathcal{R}_{M-i}$, so $v_i \in \pi(\Psi(v_{i-1})) \cap \mathcal{R}_{M-1-i}$ and it follows from the definition of $\mathcal{R}_{M-i}$ that $v_{i-1} \in \mathcal{R}_{M-i}$. This contradicts the minimality of i and concludes the proof of our claim. It follows that, for any $i \in \{0, \ldots, M-1\}$, we have

[7] We recall that $\mathcal{V} \subset \mathbb{Z}_{d,\ell}$ by Lemma 24, so that any element of $\mathcal{V}$ and, hence, of the finite set $\mathcal{E} \subset \mathcal{V}$ is of the form $\frac{a}{d\ell^h}$ for some $a \in \mathbb{Z}$ and $h \in \mathbb{Z}_{\geq 0}$.

$v_{i+1} \neq v_i$ because $v_{i+1} \in \mathcal{R}_{M-1-i}$ by construction and $v_i \notin \mathcal{R}_{M-1-i}$ by the previous claim. Then, Lemma 34 applied to $w = v_{i+1}$ and $w' = v_i$ ensures that $v_{i+1} > v_i$. In conclusion, $v_0, v_1, \ldots, v_M \in \mathcal{V}$ satisfy the following properties:

- $v_M \in \mathcal{R}_0 = \mathcal{E} \cup -\mathcal{S}(L)$;
- $v_{i+1} \in \pi(\Psi(v_i))$ for all $i \in \{0, \ldots, M-1\}$;
- $v_0 < v_1 < \cdots < v_M$.

It follows from Lemma 28 that

$$M \leq (n+1)\frac{v_M + \mu_K}{\tau} + h(v_M)\,.$$

Since $v_M \leq N$ and $h(v_M) \leq H$ and since M is an integer, we have $M \leq c$. This concludes the proof. $\square$

We note the following results for further use.

Lemma 59. *With the notations of Theorem 56,* $\max \mathcal{R} = \max \mathcal{E} \cup -\mathcal{S}(L)$.

Proof. The inequality $\max \mathcal{R} \geq \max \mathcal{E} \cup -\mathcal{S}(L)$ follows from the fact that $\mathcal{E} \cup -\mathcal{S}(L) = \mathcal{R}_0 \subset \mathcal{R}$. Proving the converse inequality $\max \mathcal{R} \leq \max \mathcal{E} \cup -\mathcal{S}(L)$ is equivalent to proving that, for all $i \in \mathbb{Z}_{\geq 0}$,

$$\max \mathcal{R}_i \leq \max \mathcal{E} \cup -\mathcal{S}(L). \tag{81}$$

Let us prove this by induction on i. The inequality (81) is obvious when $i = 0$ since $\mathcal{R}_0 = \mathcal{E} \cup -\mathcal{S}(L)$. Suppose that the inequality (81) is proved for some $i \in \mathbb{Z}_{\geq 0}$. Let $v \in \mathcal{R}_{i+1}$. By definition of $\mathcal{R}_{i+1}$, there exists $v' \in \mathcal{R}_i$ such that $v' \in \pi(\Psi(v))$. By induction hypothesis, $v' \leq \max \mathcal{E} \cup -\mathcal{S}(L)$. By Lemma 15, $v = \min \pi(\Psi(v))$, so $v \leq v' \leq \max \mathcal{E} \cup -\mathcal{S}(L)$. Therefore, $\max \mathcal{R}_{i+1} \leq \max \mathcal{E} \cup -\mathcal{S}(L)$. This concludes the induction. $\square$

Proposition 60. *We make the same assumptions and use the same notations as in Proposition 58. Let* $\check{\tau}$ *be a positive lower bound of* τ*. Then, the sequence* $(\mathcal{R}_i)_{i\geq 0}$ *can also be recursively computed as follows:*

$$\mathcal{R}_0 = (\mathcal{E} \cup -\mathcal{S}(L)) \cap \mathcal{V}_M$$

and, for all $i \geq 0$*,*

$$\mathcal{R}_{i+1} = \bigcup_{(\ell^\alpha, \beta) \in \mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}_M \tag{82}$$

where

$$M = (n+1)\left(\left\lfloor (n+1)\frac{N + \mu_K}{\check{\tau}} \right\rfloor + H\right).$$

Proof. We have seen in Theorem 56 that the sequence $(\mathcal{R}_i)_{i\geq 0}$ can be recursively computed as follows:

$$\mathcal{R}_0 = \mathcal{E} \cup -\mathcal{S}(L)$$

and, for all $i \geq 0$,

$$\mathcal{R}_{i+1} = \bigcup_{(\ell^\alpha, \beta) \in \mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta).$$

Intersecting with $\mathcal{V}_M$, we obtain:

$$\mathcal{R}_0 \cap \mathcal{V}_M = (\mathcal{E} \cup -\mathcal{S}(L)) \cap \mathcal{V}_M$$

and, for all $i \geq 0$,

$$\mathcal{R}_{i+1} \cap \mathcal{V}_M = \bigcup_{(\ell^\alpha,\beta)\in\mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}_M.$$

Given this formula, in order to prove the Proposition, it is clearly sufficient to prove that, for all $i \geq 0$, we have $\mathcal{R}_i \subset \mathcal{V}_M$. Let us prove this.

We claim that it is sufficient to prove that $\mathcal{R}_d \subset \mathcal{V}_M$ where

$$d = \left\lfloor (n+1)\frac{N+\mu_K}{\breve{\tau}} \right\rfloor + H.$$

Indeed, Proposition 58 guarantees that, for all $i \in \mathbb{Z}_{\geq 0}$, $\mathcal{R}_i \subset \mathcal{R}_c = \mathcal{R}$ where

$$c = \left\lfloor (n+1)\frac{N+\mu_K}{\tau} \right\rfloor + H.$$

But, since $\breve{\tau}$ is a positive lower bound of τ, we have

$$d = \left\lfloor (n+1)\frac{N+\mu_K}{\breve{\tau}} \right\rfloor + H \geq c.$$

So, $\mathcal{R}_c = \mathcal{R}_d = \mathcal{R}$ and, for all $i \in \mathbb{Z}_{\geq 0}$, $\mathcal{R}_i \subset \mathcal{R}_c = \mathcal{R}_d$. This justifies our claim.

We now claim that, in order to conclude the proof, it is sufficient to prove that, for all $v \in \mathcal{R}_d$, we have

(1) $v \leq N$;
(2) $h(v) \leq H + nd$.

Indeed, Proposition 22 ensures that, for all $v \in \mathcal{R}_d$, we have

$$v \in \mathcal{V}_{\left\lfloor (n+1)\frac{v+\mu_K}{\tau} + h(v) \right\rfloor}.$$

But, if (1) and (2) are true, then we have, for all $v \in \mathcal{R}_d$,

$$\left\lfloor (n+1)\frac{v+\mu_K}{\tau} + h(v) \right\rfloor \leq \left\lfloor (n+1)\frac{N+\mu_K}{\breve{\tau}} + H + nd \right\rfloor = (n+1)d = M.$$

So $\mathcal{R}_d \subset \mathcal{V}_M$. This proves our claim.

In order to complete the proof, it only remains to prove (1) and (2). The inequality (1) is a direct consequence of Lemma 59 and of our choice of N. To justify inequality (2), we prove more generally that, for all $i \in \mathbb{Z}_{\geq 0}$, for all $v \in \mathcal{R}_i$, $h(v) \leq H + ni$. We proceed by induction on i. The base case $i = 0$ is true by our choice of H. We now assume that, for some $i \in \mathbb{Z}_{\geq 0}$, we have, for all $v \in \mathcal{R}_i$, $h(v) \leq H + ni$. Consider $v \in \mathcal{R}_{i+1}$. By definition of the sequence $(\mathcal{R}_i)_{i\geq 0}$, the set $\pi(\Psi(v)) \cap \mathcal{R}_i$ is nonempty (see (77)); let w be in this intersection. By Lemma 24, we have $h(v) \leq h(w) + n$. But, by the inductive hypothesis, we have $h(w) \leq H + ni$. So $h(v) \leq H + ni + n = H + n(i+1)$. This concludes the induction. □

8. Answer to Question 1

In this section, we consider a Mahler operator

$$L = a_n\phi_\ell^n + a_{n-1}\phi_\ell^{n-1} + \cdots + a_0$$

with coefficients $a_0, \ldots, a_n \in \mathbf{K}[z]$ such that $a_0 a_n \neq 0$.

8.1. **The algorithm.**

Algorithm 61.

Input: L a Mahler operator with coefficients in $\mathbf{K}[z]$, $\mathcal{E}$ a finite subset of $\mathbb{Q}$.
Output: the image under $\bullet_{|\mathcal{E}}$ of a basis of $\mathrm{Sol}(L, \mathscr{H})$; it is a generating family of the $\mathbf{K}$-vector space made of the $f \in \mathscr{H}_{|\mathcal{E}}$ for which there exists a solution $\widetilde{f} \in \mathscr{H}$ of L such that $\widetilde{f}_{|\mathcal{E}} = f$ and even a basis if $-\mathcal{S}(L) \subset \mathcal{E}$.

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set $\mathcal{R}_{-1} = \emptyset$
set $\check{\tau}$ =Lower_Bound_τ (L) (see Algorithm 44)
let H be the least integer such that $\ell^H \mathcal{E} \subset \frac{1}{d}\mathbb{Z}$
set $N = \max \mathcal{E} \cup -\mathcal{S}(L)$
set $M = (n+1)\left(\left\lfloor (n+1)\frac{N+\mu_K}{\check{\tau}} \right\rfloor + H\right)$
compute $\mathcal{R}_0 = (\mathcal{E} \cup -\mathcal{S}(L)) \cap \mathcal{V}_M$
set $i = 0$
while $\mathcal{R}_i \neq \mathcal{R}_{i-1}$
  compute $\mathcal{R}_{i+1} = \bigcup_{(\ell^\alpha, \beta) \in \mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}_M$
  increment i by 1
end while
set $\mathcal{R} = \mathcal{R}_i$;
compute a basis $(f_1, \ldots, f_t)$ of the $\mathbf{K}$-vector space $\mathcal{C}_{\mathcal{R}}$
return $(\bullet_{|\mathcal{E}}(f_1), \ldots, \bullet_{|\mathcal{E}}(f_t))$.

Theorem 62. *Algorithm 61 answers Question 1 by the positive.*

Proof. Let us first consider the computability issues. We can compute $\mathcal{V}_M$ using the recursive formula from Theorem 17. Then, we can compute $\mathcal{R}_0 = (\mathcal{E} \cup -\mathcal{S}(L)) \cap \mathcal{V}_M$ because $\mathcal{E} \cup -\mathcal{S}(L)$ and $\mathcal{V}_M$ are explicit finite sets. We can compute $\mathcal{R}_{i+1}$ from $\mathcal{R}_i$ with the formula $\mathcal{R}_{i+1} = \bigcup_{(\ell^\alpha, \beta) \in \mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}_M$ because $\mathcal{P}(L)$, $\mathcal{V}_M$ and, for any $(\ell^\alpha, \beta) \in \mathcal{P}(L)$, $\ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta)$ are explicit finite sets. Proposition 60 and Theorem 56 guaranty that $\mathcal{R}_i = \mathcal{R}_{i-1}$ for some $i \in \mathbb{Z}_{\geq 1}$, so the "while" loop will stop after finitely many steps. Once $\mathcal{R}$ has been calculated, computing a basis $(f_1, \ldots, f_t)$ of $\mathcal{C}_{\mathcal{R}}$ amounts to compute a basis of solutions of an explicit system of linear equations. Indeed one can compute explicit linear maps $(F_\delta : \mathbf{K}^{\mathcal{R}} \to \mathbf{K})_{\delta \in \psi(\mathcal{R})}$ such that, for any $f = \sum_{\gamma \in \mathcal{R}} f_\gamma z^\gamma \in \mathscr{H}_{|\mathcal{R}}$,

$$L(f) = \sum_{\delta \in \psi(\mathcal{R})} F_\delta((f_\gamma)_{\gamma \in \mathcal{R}}) z^\delta + \text{terms whose support is disjoint from } \psi(\mathcal{R}).$$

So, f belongs to $\mathcal{C}_{\mathcal{R}}$ if and only if, for all

$$F_\delta((f_\gamma)_{\gamma \in \mathcal{R}}) = 0, \quad \text{for all } \delta \in \psi(\mathcal{R}). \tag{83}$$

Finding a basis of $\mathcal{C}_{\mathcal{R}}$ amounts to finding a basis of solutions of this system of linear equations. This can be done algorithmically.

Let us now justify that this algorithm returns the correct output, namely the image under $\bullet_{|\mathcal{E}}$ of a basis of $\mathrm{Sol}(L, \mathscr{H})$.

It follows from Theorem 17 that $\mathrm{Sol}(L, \mathscr{H}) \subset \mathscr{H}_{|\mathcal{V}}$. Since $\mathcal{R}$ satisfies $\star_{\mathcal{R}_0,\mathcal{V}}$ by Proposition 60 and Theorem 56, it follows from Theorem 46 that the map $\bullet_{|\mathcal{R}} : \mathscr{H} \to \mathscr{H}_{|\mathcal{R}}$ induces an isomorphism between $\mathrm{Sol}(L, \mathscr{H})$ and $\mathcal{C}_{\mathcal{R}}$. So, $(\bullet_{|\mathcal{R}}^{-1}(f_1), \ldots, \bullet_{|\mathcal{R}}^{-1}(f_t))$ is a basis of $\mathrm{Sol}(L, \mathscr{H})$. But, for all $i \in \{1, \ldots, t\}$, after setting $g_i = \bullet_{|\mathcal{R}}^{-1}(f_i)$, which is an element of $\mathrm{Sol}(L, \mathscr{H})$, we have $\mathrm{supp}(g_i) \cap \mathcal{E} \subset \mathrm{supp}(g_i) \cap \mathcal{R}$ and, hence,

$$\bullet_{|\mathcal{E}}(\bullet_{|\mathcal{R}}^{-1}(f_i)) = \bullet_{|\mathcal{E}}(g_i) = \bullet_{|\mathcal{E}}(\bullet_{|\mathcal{R}}(g_i)) = \bullet_{|\mathcal{E}}(f_i).$$

Thus,

$$(\bullet_{|\mathcal{E}}(f_1), \ldots, \bullet_{|\mathcal{E}}(f_t)) = (\bullet_{|\mathcal{E}}(\bullet_{|\mathcal{R}}^{-1}(f_1)), \ldots, \bullet_{|\mathcal{E}}(\bullet_{|\mathcal{R}}^{-1}(f_t)))$$

is the image under $\bullet_{|\mathcal{E}}$ of a basis of $\mathrm{Sol}(L, \mathscr{H})$. Thus, $(\bullet_{|\mathcal{E}}(f_1), \ldots, \bullet_{|\mathcal{E}}(f_t))$ is a generating family of the $\mathbf{K}$-vector space $\bullet_{|\mathcal{E}}(\mathrm{Sol}(L, \mathscr{H}))$, which is nothing but the $\mathbf{K}$-vector space made of the $f \in \mathscr{H}_{|\mathcal{E}}$ for which there exists a solution $\widetilde{f} \in \mathscr{H}$ of L such that $\widetilde{f}_{|\mathcal{E}} = f$. Last, if $-\mathcal{S}(L) \subset \mathcal{E}$, then the restriction of $\bullet_{|\mathcal{E}}$ to $\mathrm{Sol}(L, \mathscr{H})$ is injective as a consequence of Corollary 13 and, hence, $(\bullet_{|\mathcal{E}}(f_1), \ldots, \bullet_{|\mathcal{E}}(f_t))$ is a basis of the $\mathbf{K}$-vector space $\bullet_{|\mathcal{E}}(\mathrm{Sol}(L, \mathscr{H}))$. □

8.2. **On the complexity of Algorithm 61.** In this section, by "complexity" we mean the number of basic operations ($+$, $-$, $\times$, $\div$) in $\mathbf{K}$ and comparisons in $\mathbb{Q} \cup \{-\infty, +\infty\}$ performed by an algorithm.

To estimate the complexity of Algorithm 61, we shall suppose that $n \geq 2$. Indeed, if $n = 1$, then (1) has a nonzero solution $f \in \mathscr{H}$ if and only if the coefficient of $z^{\mathrm{val}(a_0)}$ in a_0 is the opposite of the coefficient of $z^{\mathrm{val}(a_1)}$ in a_1; in this case, we have

$$f = \lambda z^{\frac{\mathrm{val}(a_1)-\mathrm{val}(a_0)}{\ell-1}} \prod_{k=0}^{\infty} \frac{-a_1(z^{\ell^k})z^{-\ell^k \mathrm{val}(a_1)}}{a_0(z^{\ell^k})z^{-\ell^k \mathrm{val}(a_0)}}$$

for some $\lambda \in \mathbf{K} \setminus \{0\}$. Thus, there is no need using Algorithm 61.

Proposition 63. *Suppose that $n \geq 2$. Let τ be defined by (38) and $\breve{\tau} > 0$ be a lower bound on τ. Let N be an integer and let*

$$\mathcal{E} = \mathcal{E}_N = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\}, \max\{|a|, |b|\} \leq N \right\}$$

Suppose that N is large enough so that $\mathcal{S}(L) \subset \mathcal{E}$. Then, Algorithm 61 has complexity

$$\mathcal{O}\left((\delta n)^{3n^2 N/\breve{\tau}} \right) \tag{84}$$

when one does not take into account the complexity of computing $\breve{\tau}$.

Remark 64. *1. The complexity of the algorithm in Theorem 62 depends strongly on the lower bound $\breve{\tau}$ computed by Algorithm 44.*

2. Of course, the complexity of this algorithm depends on the choice of the integer ℓ. In (84), this dependency is hidden in the parameter $\breve{\tau}$ which is bounded from above by ℓ^n.

3. In comparison, the algorithm given in [CDDM18] to find Puiseux solutions has complexity $\tilde{\mathcal{O}}(n^2 N d \ell^n)$, where d is defined as in Section 4.

Proof. We note that, with the notation of the algorithm, we have $H \leq \lceil \log N / \log \ell \rceil$.

Computation of the Newton polygon. One can compute the set $\mathcal{P}(L)$ in $\mathcal{O}(\sharp\mathcal{P}(L))$ operations and the set

$$\{(j, \operatorname{val} a_j(z)) \mid j \in \{0, \dots, n\}\}$$

with the same complexity. Then, one can compute the set $\mathcal{S}(L)$ of slopes of $\mathcal{N}(L)$ and the endpoints of these slopes by performing $\mathcal{O}(n)$ comparisons and operations. Furthermore, one may return the set of slopes as an ordered list of rational numbers with the same complexity.

Computation of $\mathcal{V}_M$. The set $\mathcal{V}_0 = -\mathcal{S}(L)$ has K elements. This set can be computed in $\mathcal{O}(K)$ operations, once $\mathcal{S}(L)$ is known. Then, for any i, the set $\mathcal{V}_i$ has at most $(\sharp\mathcal{P}(L))^i K$ elements. Suppose that the set $\mathcal{V}_i$ has been computed for some i and that it is given as an ordered list of rational numbers. Let us compute the ordered list of all elements of $\mathcal{V}_{i+1}$. Fix a point $p = (\ell^\alpha, \beta) \in \mathcal{P}(L)$. Then, since $\mathcal{V}_i$ is given as an ordered list, one may compute the ordered list of elements of the set

$$\mathcal{V}_i(p) = \{\pi(v\ell^\alpha + \beta) \mid v \in \mathcal{V}_i\}$$

in $\mathcal{O}((\sharp\mathcal{P}(L))^i K)$. Thus, the computation of the $\sharp\mathcal{P}(L)$ lists $\mathcal{V}_i(p)$, $p \in \mathcal{P}(L)$, requires $\mathcal{O}((\sharp\mathcal{P}(L))^{i+1} K)$ operations. Given k ordered lists containing m elements each, one may order the union of the k lists in $\mathcal{O}(km\log(k))$ operations. Thus, once the ordered lists of elements of $\mathcal{V}_i(p)$, $p \in \mathcal{P}(L)$, are computed, the ordered list of elements of

$$\mathcal{V}_{i+1} = \bigcup_{p \in \mathcal{P}(L)} \mathcal{V}_i(p)$$

can be computed in $\mathcal{O}((\sharp\mathcal{P}(L))^{i+1} K \log(\sharp P(L)))$. *In fine*, the computation of the ordered list of elements of $\mathcal{V}_M$ can be performed with

$$\mathcal{O}((\sharp\mathcal{P}(L))^M K \log(\sharp P(L))) \tag{85}$$

operations. Furthermore, one has

$$\sharp\mathcal{V}_M \leq (\sharp\mathcal{P}(L))^M K. \tag{86}$$

Computation of $\mathcal{R}$. We now have an ordered list of all elements of $\mathcal{V}_M$. One may compute an ordered list of all elements of $\mathcal{E}$ in $\mathcal{O}(\sharp\mathcal{E}\log(\sharp\mathcal{E}))$. Two ordered lists of rational numbers being given, one can compute the ordered list of elements belonging to both lists making a number of comparisons at most equal to the maximum of the size of these lists. Thus, one may compute the ordered list of elements of $\mathcal{R}_0 = \mathcal{E} \cap \mathcal{V}_M$ by making $\mathcal{O}(\sharp\mathcal{E} + \sharp\mathcal{V}_M)$ comparisons. Suppose that the ordered list of elements of $\mathcal{R}_i$ has been computed, for some integer $i \geq 0$. Note that $\sharp\mathcal{R}_i \leq \sharp\mathcal{E}(\sharp\mathcal{P}(L))^i$. Then, for each $p = (\ell^\alpha, \beta) \in \mathcal{P}(L)$ one can compute the ordered list of elements of

$$\ell^\alpha(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}_M$$

in $\mathcal{O}(\sharp\mathcal{E}(\sharp\mathcal{P}(L))^i + \sharp\mathcal{V}_M)$ operations. Once these $\sharp\mathcal{P}(L)$ lists are stored, one may compute the ordered list of elements of $\mathcal{R}_{i+1}$ by making

$$\mathcal{O}(\sharp\mathcal{E}(\sharp\mathcal{P}(L))^{i+1} \log(\sharp\mathcal{P}(L)))$$

comparisons. In fine, the total complexity of the computation of $\mathcal{R} = \mathcal{R}_c$ is in

$$\mathcal{O}(\sharp\mathcal{E}(\sharp\mathcal{P}(L))^c \log(\sharp\mathcal{P}(L)) + c\sharp\mathcal{P}(L)\sharp\mathcal{V}_M) \tag{87}$$

Furthermore, we have

$$\sharp\mathcal{R} \leq \sharp\mathcal{E}(\sharp\mathcal{P}(L))^c \,. \tag{88}$$

Computation of a basis of $\mathcal{C}_\mathcal{R}$. Now that the ordered list of elements of $\mathcal{R}$ has been computed, one may compute the ordered list of elements of $\psi(\mathcal{R})$ with $\mathcal{O}(\sharp\mathcal{R})$ operations. The coefficients of the system of linear equations defining $\mathcal{C}_\mathcal{R}$ can be computed with $\mathcal{O}(\sharp\mathcal{P}(L)\sharp\mathcal{R})$ operations. Solving a linear system with m indeterminates necessitates $\mathcal{O}(m^\Theta)$ operations, for some $\Theta < 3$ (for example, one may take $\Theta = \log_2(7) \simeq 2,81$; see [BCG+17]). Thus, once the coefficients of the system of linear equations is known, one may compute a basis of $\mathcal{C}_\mathcal{R}$ in

$$\mathcal{O}((\sharp\mathcal{R})^\Theta) \tag{89}$$

operations.

Total complexity of the algorithm. Once a basis of $\mathcal{C}_\mathcal{R}$ is known, the complexity of the end of the algorithm is negligible with respect to the number of operations performed so far. Combining (85), (86), (87), (88) and (89), the complexity of this algorithm is

$$\mathcal{O}(\sharp\mathcal{E}^\Theta(\sharp\mathcal{P}(L))^{c\Theta} + c\sharp\mathcal{P}(L)^{M+1}K) \,. \tag{90}$$

Furthermore, we have the following bounds

$$\left\{\begin{array}{rcl} \sharp\mathcal{P}(L) & \leq & (\delta+1)(n+1)\,, \\ \sharp\mathcal{E} & \leq & 2N^2\,, \\ \mu_K & \leq & N\,, \\ c & \leq & 2(n+1)N\check{\tau}^{-1} + \log N\,, \\ M & \leq & 2(n+1)^2N\check{\tau}^{-1} + (n+1)\log N\,, \\ K & \leq & n\,, \\ \Theta & < & n+1\,. \end{array}\right.$$

Thus, the second term in (90) dominates the first one. It follows that the complexity of the algorithm is

$$\mathcal{O}\left(\left(2(n+1)N\check{\tau}^{-1} + \log N\right)\left((\delta+1)(n+1)\right)^{2(n+1)^2N\check{\tau}^{-1}+(n+1)\log N+1}n\right)$$

Now, the result follows from the fact that the quantity above is in

$$\mathcal{O}\left((\delta n)^{3n^2N/\check{\tau}}\right).$$

□

9. An example : the Rudin-Shapiro Mahler equation

Consider the following 2-Mahler equation:

$$-2zy(z^4) + (z-1)y(z^2) + y(z) = 0\,. \tag{91}$$

One can prove that:

i) up to a multiplicative constant, the only solution of (91) in $\mathscr{H}$ is actually a power series, namely the generating series of the Rudin-Shapiro sequence, see [AS03] for instance;
ii) (91) has another nonzero solution of the form $fe_{-\frac{1}{2}}$ where $f \in \mathscr{H}$ and where $e_{-\frac{1}{2}}$ satisfies $\phi_2(e_{-\frac{1}{2}}) = -\frac{1}{2}e_{-\frac{1}{2}}$.

We won't be proving property i) here, as that would take us too far from our objective to illustrate Algorithm 61. Property ii) could be proved using [Roq22] but we will give another proof based on Algorithm 61, which has the advantage of also giving as many coefficients of f as we like.

Let us first note that $fe_{-\frac{1}{2}}$ is a solution of (91) if and only if f is a solution of the 2-Mahler equation

$$zy(z^4) + (z-1)y(z^2) - 2y(z) = 0, \tag{92}$$

which is nothing but the 2-Mahler equation associated to the 2-Mahler operator

$$L = z\phi_2^2 + (z-1)\phi_2 - 2 \tag{93}$$

defined by (14). In what follows, we will run Algorithm 61 for this L and for the set $\mathcal{E}$ defined by

$$\mathcal{E}_8 := \left\{ \frac{a}{b} \in \mathbb{Q} \mid \max\{|a|,|b|\} \le 8 \right\}. \tag{94}$$

We will see that the output of this algorithm is

$$\begin{aligned} &z^{-\frac{1}{2}} - 2z^{-\frac{1}{4}} + 4z^{-\frac{1}{8}} \\ &\quad - \frac{1}{3} + z^{\frac{1}{2}} - 2z^{\frac{3}{4}} + 4z^{\frac{7}{8}} - \frac{5}{6}z + z^{\frac{3}{2}} - 2z^{\frac{7}{4}} + \frac{11}{12}z^2 - z^{\frac{5}{2}} \\ &\qquad - \frac{5}{12}z^3 + z^{\frac{7}{2}} - \frac{23}{24}z^4 + \frac{13}{24}z^5 - \frac{7}{24}z^6 - \frac{5}{24}z^7 - \frac{1}{48}z^8. \end{aligned} \tag{95}$$

Since $-\mathcal{S}(L) \subset \mathcal{E}$, this shows that the $\mathbf{K}$-vector space $\bullet_{|\mathcal{E}}(\mathrm{Sol}(L,\mathscr{H}))$ has dimension 1 and is generated by (95). Moreover, since the restriction of $\bullet_{|\mathcal{E}}$ to $\mathrm{Sol}(L,\mathscr{H})$ is injective by Corollary 13, this proves that

$$\mathrm{Sol}(L,\mathscr{H}) = \mathbb{C}f$$

for some $f \in \mathscr{H}$ such that $\bullet_{|\mathcal{E}}(f) =$(95). Therefore, $fe_{-\frac{1}{2}}$ is a solution of (91) such that $\bullet_{|\mathcal{E}}(f) =$(95). This justifies property ii) above and gives moreover the value of the coefficients of f corresponding to the indices in $\mathcal{E}$.

Let us take a close look at how Algorithm 61 works when we take as input the operator L given by (14) and the set $\mathcal{E}$ given by (94). This is done in section 9.5, after some preliminaries.

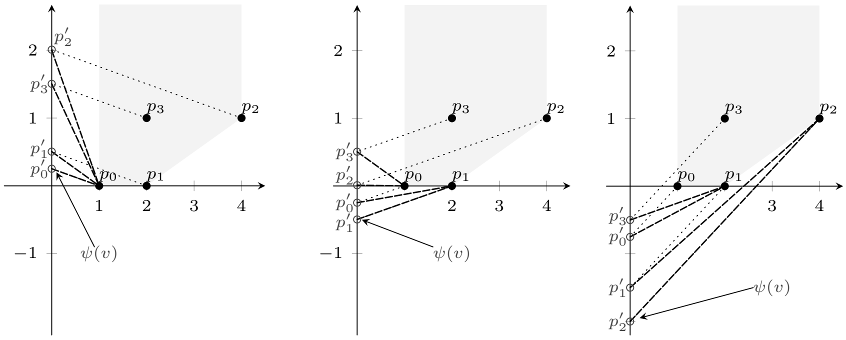
9.1. **Newton polygon and slopes of L.** The Newton polygon $\mathcal{N}(L)$ of L is the lower convex hull of the set

$$\mathcal{P}(L) = \{(1,0),(2,0),(2,1),(4,1)\}.$$

We have

$$\mathcal{S}(L) = \{\mu_1,\mu_2\} \text{ with } \mu_1 = 0 \text{ and } \mu_2 = \frac{1}{2}.$$

A common denominator of the slopes of L is thus $d = 2$.



(A) Case $v = 1/4$ (B) Case $v = -1/4$ (C) Case $v = -3/4$

FIGURE 2. This figure is relative to the operator L given by (14). We have $\mathcal{P}(L) = \{p_0, p_1, p_2, p_3\}$. The Newton polygon $\mathcal{N}(L)$ is the shaded area. Its vertices are p_0, p_1 and p_2 and we have $\mathcal{S}(L) = \{0, 1/2\}$. In each subfigure, we consider a specific $v \in \mathbb{Q}$. The point p'_k is the projection of p_k along a line of slope $-v$ onto the y-axis; so, the dotted segments have slope $-v$ and $\Psi(v) = \{p'_0, p'_1, p'_2, p'_3\}$. The dashed segment with left extremity p'_k is the segment with lowest slope among those linking p'_k to an element of $\mathcal{P}(L)$. The slope of this segment is thus the opposite of $\pi(q_k)$ where q_k is the ordinate of p'_k and, hence, $\pi(\Psi(v))$ is the set of the opposite of the slopes of the four dashed segments.

The vertices, ordered by increasing abscissa, of the polygon $\mathcal{N}(L)$ are

$$p_0 = (1, 0),\ p_1 = (2, 0) \text{ and } p_2 = (4, 1).$$

For any $k \in \{0, 1, 2\}$, we have

$$p_k = (\ell^{\alpha_k}, \beta_k) = (\ell^{\alpha_k}, \operatorname{val} a_{\alpha_k})$$

with

$$\alpha_0 = 0, \beta_0 = 0, \alpha_1 = 1, \beta_1 = 0, \alpha_2 = 2, \beta_2 = 1.$$

The set $\mathcal{P}(L)$, the Newton polygon $\mathcal{N}(L)$ and the vertices p_k are represented in Figure 2.

9.2. **The maps π and Ψ.** Straightforward calculations show that

$$\Psi(v) = \{v, 2v, 2v + 1, 4v + 1\} \text{ and } \pi(q) = \begin{cases} q/2 & \text{if } -1 \leq q < 0, \\ q & \text{if } q \geq 0. \end{cases}$$

We do not need to specify $\pi(q)$ when $q < -1$.

9.3. **The sets $\mathcal{V}_i$.** One can compte as many $\mathcal{V}_i$ as necessary using their recursive definition. For instance,

$$
\begin{array}{rcl}
\mathcal{V}_0 & = & -\mathcal{S}(L) = \left\{-\frac{1}{2}, 0\right\}, \\
\mathcal{V}_1 & = & \pi\left(\Psi\left(-\frac{1}{2}\right) \cup \Psi(0)\right) = \pi\left(\left\{-1, -\frac{1}{2}, 0, 1\right\}\right) = \left\{-\frac{1}{2}, -\frac{1}{4}, 0, 1\right\}, \\
\mathcal{V}_2 & = & \pi\left(\Psi\left(-\frac{1}{2}\right) \cup \Psi\left(-\frac{1}{4}\right) \cup \Psi\left(0\right) \cup \Psi(1)\right) \\
 & = & \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, 0, \frac{1}{2}, 1, 2, 3, 5\right\} \\
\cdots & \cdots & \cdots
\end{array}
$$

Remark 65. *In this very peculiar example, we could prove that*

$$\mathcal{V} = \left\{k - \frac{1}{2^n} \mid k, n \in \mathbb{Z}_{\geq 0},\ (k, n) \neq (0, 0)\right\}.$$

As this property will not be used, we only briefly indicate how to prove the most useful inclusion, namely the inclusion of $\mathcal{V}$ in the right-hand side of the latter equality, and leave the details and proof of the other inclusion to the reader. Let us denote this right-hand side by $\mathcal{W}$. One can easily check that $-\mathcal{S}(L) \subset \mathcal{W}$ and that, for any $w \in \mathcal{W}$, $\pi(\Psi(w)) \subset \mathcal{W}$. Given the definition of $\mathcal{V}$, this clearly implies that $\mathcal{V} \subset \mathcal{W}$.

When running Algorithm 61, we will first need to call Algorithm 44 in order to calculate a positive lower bound $\breve{\tau}$ on τ, which is defined by (38).

9.4. **Computation of a positive lower bound on τ.** We will now explain how Lower_Bound_$\tau(L)$ described in Algorithm 44 runs to compute a lower bound $\breve{\tau}$ on τ. It takes the following steps:

(1) it computes a lower bound

$$\breve{\epsilon}_2 = \mathrm{LB\_}\epsilon\mathrm{\_p}(L, 2, (), -\mu_2) {=} \mathrm{LB\_}\epsilon\mathrm{\_p}(L, 2, (), -\frac{1}{2})$$

on $\epsilon(-\mu_2) = \epsilon(-\frac{1}{2})$;

(2) it computes a lower bound

$$\breve{\epsilon}_1 = \mathrm{LB\_}\epsilon\mathrm{\_p}(L, 1, (\breve{\epsilon}_2), -\mu_1) {=} \mathrm{LB\_}\epsilon\mathrm{\_p}(L, 1, (\breve{\epsilon}_2), 0)$$

on $\epsilon(-\mu_1) = \epsilon(0)$;

(3) it returns

$$\breve{\tau} = \min\{\breve{\epsilon}_1, \breve{\epsilon}_2, \frac{1}{2 \cdot 2^2}\};$$

where we have written LB_ϵ_p for Lower_Bound_ϵ_param defined in Algorithm 42, in order to avoid heavy notations. These three steps are detailed in the following three sections.

9.4.1. *Execution of LB_ϵ_p$(L, 2, (), -\mu_2)$ =LB_ϵ_p$(L, 2, (), -\frac{1}{2})$.* We are in the situation starting in row 4 of Algorithm 42 since $-\frac{1}{2} = -\mu_2$. Thus, LB_ϵ_p$(L, 2, (), -\frac{1}{2})$ returns

$$\min\left(\mathcal{V}_1 \setminus \left\{-\frac{1}{2}\right\}\right) + \frac{1}{2} = -\frac{1}{4} + \frac{1}{2} = \frac{1}{4}.$$

This latter value $\frac{1}{4}$ is stored in $\breve{\epsilon}_2$.

9.4.2. *Execution of LB_ϵ_p$(L, 1, (\breve{\epsilon}_2), -\mu_1)$ =LB_ϵ_p$(L, 1, (\frac{1}{4}), 0)$.* It takes the following steps.

(1) Since $0 = -\mu_1$, we are in the situation starting in row 16 of Algorithm 42 with $k = 2$.
(2) For any $w' \in \Delta(-\mu_1) = \Delta(0) = \{-\frac{1}{2}, -\frac{1}{4}\}$, it computes

$$m_{w'} = \mathrm{LB\_}\epsilon\mathrm{\_p}(L, 1, (\frac{1}{4}), w').$$

(3) LB_ϵ_p$(L, 1, (\frac{1}{4}), -\frac{1}{2})$ returns $\frac{1}{4}$ after calculations similar to those described in 9.4.1.
(4) LB_ϵ_p$(L, 1, (\frac{1}{4}), -\frac{1}{4})$ runs as follows.
   (a) Since $-\frac{1}{4} \in]-\frac{1}{2}, 0[=]-\mu_2, -\mu_1[$, we are in the situation starting in row 12 of Algorithm 42 with $k = 2$. So, LB_ϵ_p$(L, 1, (\frac{1}{4}), -\frac{1}{4})$ calls LB_ϵ_i $(L, 1, (\frac{1}{4}), -\frac{1}{4})$, where we have written LB_ϵ_i for Lower_Bound_ϵ_interval defined in Algorithm 43, in order to avoid heavy notations.
   (b) LB_ϵ_i $(L, 1, (\frac{1}{4}), -\frac{1}{4})$ runs as follows.
      (i) Since $-\frac{1}{4} \geq -\mu_2 + \breve{\epsilon}_2 = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$, we are in the situation starting in row 10 of Algorithm 43. Therefore, for each $v \in \Delta(-\frac{1}{4}) = \{-\frac{1}{2}, -\frac{3}{4}, -\frac{3}{8}\}$, it computes $\boldsymbol{b}_v =$ LB_ϵ_i$(L, 1, (\frac{1}{4}), v)$.
         (A) LB_ϵ_i$(L, 1, (\frac{1}{4}), -\frac{3}{4})$ returns $(-\frac{3}{4}, \frac{1}{4})$ because $-\frac{3}{4} < -\frac{1}{2} = -\mu_2$ so we are in the situation starting with row 3 of Algorithm 43 so it calls LB_ϵ_p$(L, 2, (), -\frac{3}{4})$ which returns $-\mu_2 - (-\frac{3}{4}) = -\frac{1}{2} + \frac{3}{4} = \frac{1}{4}$ because we are in the situation starting in row 1 of Algorithm 42.
         (B) LB_ϵ_i$(L, 1, (\frac{1}{4}), -\frac{1}{2})$ returns $(-\frac{1}{2}, \frac{1}{4})$ because $-\frac{1}{2} < -\mu_2 + \breve{\epsilon}_2 = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$ and $-\frac{1}{2} \geq -\mu_2 = -\frac{1}{2}$ so we are in the situation starting with row 5 of Algorithm 43 and $-\mu_2 + \breve{\epsilon}_2 - (-\frac{1}{2}) = -\frac{1}{2} + \frac{1}{4} + \frac{1}{2} = \frac{1}{4}$.
         (C) LB_ϵ_i$(L, 1, (\frac{1}{4}), -\frac{3}{8})$ returns $(-\frac{3}{8}, \frac{1}{8})$ because $-\frac{3}{8} < -\mu_2 + \breve{\epsilon}_2 = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$ and $-\frac{3}{8} \geq -\mu_2 = -\frac{1}{2}$ so we are in the situation starting with row 5 of Algorithm 43 and $-\mu_2 + \breve{\epsilon}_2 - (-\frac{3}{8}) = -\frac{1}{2} + \frac{1}{4} + \frac{3}{8} = \frac{1}{8}$.
      (ii) It computes the minimum m of the set (65), which is, as explained in details in Example 36, the minimum of
         - $\frac{1}{4} \times 2^{0-1} = \frac{1}{8}$,
         - $\frac{1}{4} \times 2^{1-1} = \frac{1}{4}$,
         - $\frac{1}{8} \times 2^{2-1} = \frac{1}{4}$,

- $0 - \left(-\frac{1}{4}\right) = \frac{1}{4}$,
- $\min\{\pi(\Psi(-\frac{1}{4})) \setminus \{-\frac{1}{4}\}\} + \frac{1}{4} = -\frac{1}{8} + \frac{1}{4} = \frac{1}{8}$.

We thus have $m = \frac{1}{8}$.

(iii) LB_ϵ_i$(L, 1, (\frac{1}{4}), -\frac{1}{4})$ returns $(-\frac{1}{4}, \frac{1}{8})$.

(c) LB_ϵ_p$(L, 1, (\frac{1}{4}), -\frac{1}{4})$ returns $\frac{1}{8}$.

(5) Last, it runs row 20 of Algorithm 42 and, after calculations already presented in details in Example 40, we find that LB_ϵ_p$(L, 1, (\frac{1}{4}), 0)$ returns $\frac{1}{2}$.

The latter value $\frac{1}{2}$ is stored in $\breve{\epsilon}_1$.

9.4.3. *Last step of Lower_Bound_$\tau(L)$.* Finally, Lower_Bound_$\tau(L)$ returns

$$\breve{\tau} = \min\left\{\breve{\epsilon}_1, \breve{\epsilon}_2, \frac{1}{8}\right\} = \frac{1}{8}\,.$$

9.5. **Algorithm 61.** We already computed the positive lower bound $\breve{\tau} = \frac{1}{8}$ on τ. We check that $H = 2$ and $N = 8$ by definition of $\mathcal{E}$. Then we have

$$M = (n+1)\left(\left\lfloor (n+1)\frac{N+\mu_K}{\breve{\tau}}\right\rfloor + H\right) = 618\,.$$

We could compute the set $\mathcal{V}_M$ but, in practice, we can save a few calculations by exploiting the fact that, according to Lemma 59, any element of $\mathcal{R}$ (and, hence, of any $\mathcal{R}_i$) is lower than or equal to $\max \mathcal{E} \cup -\mathcal{S}(L) = 8$. Indeed, this remark entails that the intersections with $\mathcal{V}_M$ involved in Algorithm 61 for calulating the $\mathcal{R}_i$ can be replaced by intersections with $\mathcal{V}_M \cap \mathbb{Q}_{\leq 8}$ without changing the result of the calculations, *i.e.*, we have

$$\mathcal{R}_0 = (\mathcal{E} \cup -\mathcal{S}(L)) \cap \mathcal{V}_M = (\mathcal{E} \cup -\mathcal{S}(L)) \cap \mathcal{V}_M \cap \mathbb{Q}_{\leq 8}$$

and

$$\mathcal{R}_{i+1} = \bigcup_{(\ell^\alpha,\beta)\in\mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}_M = \bigcup_{(\ell^\alpha,\beta)\in\mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_i) - \beta) \cap \mathcal{V}_M \cap \mathbb{Q}_{\leq 8}.$$

Thus, we only need to compute the set $\mathcal{V}_M \cap \mathbb{Q}_{\leq 8}$, not the whole $\mathcal{V}_M$. Let us now explain how one can compute the sets $\mathcal{V}_{i,\leq 8} = \mathcal{V}_i \cap \mathbb{Q}_{\leq 8}$. We recall that, by definition, the $(\mathcal{V}_i)_{i\geq 0}$ can be computed recursively as follows:

$$\mathcal{V}_0 = -\mathcal{S}(L)$$

and, for all $i \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{V}_{i+1} = \bigcup_{v\in\mathcal{V}_i} \pi(\Psi(v)).$$

Intersecting with $\mathbb{Q}_{\leq 8}$, we obtain:

$$\mathcal{V}_{0,\leq 8} = -\mathcal{S}(L) \cap \mathbb{Q}_{\leq 8} = -\mathcal{S}(L) \tag{96}$$

and, for all $i \in \mathbb{Z}_{\geq 0}$,

$$\mathcal{V}_{i+1,\leq 8} = \bigcup_{v\in\mathcal{V}_i} \pi(\Psi(v)) \cap \mathbb{Q}_{\leq 8}. \tag{97}$$

But, according to (25), we have, for all $v \in \mathbb{Q}$, $v = \min \pi(\Psi(v))$. So, for any $v \in \mathcal{V}_i \setminus \mathcal{V}_{i,\leq 8}$, we have $\pi(\Psi(v)) \cap \mathbb{Q}_{\leq 8} = \emptyset$ and, hence, (97) can be rewritten as follows:

$$\mathcal{V}_{i+1,\leq 8} = \bigcup_{v \in \mathcal{V}_{i,\leq 8}} \pi(\Psi(v)) \cap \mathbb{Q}_{\leq 8}. \tag{98}$$

Now, (96) and (98) allow us to recursively calculate as many $\mathcal{V}_{i,\leq 8}$ as we like. Note also that, for computing the right-hand side of (98), we only need to compute $\pi(\Psi(v))$ for $v \in \mathcal{V}_{i,\leq 8} \setminus \mathcal{V}_{i-1,\leq 8}$ because

$$\mathcal{V}_{i+1,\leq 8} = \mathcal{V}_{i,\leq 8} \cup \bigcup_{v \in \mathcal{V}_{i,\leq 8} \setminus \mathcal{V}_{i-1,\leq 8}} \pi(\Psi(v)) \cap \mathbb{Q}_{\leq 8}.$$

Using this, we may compute $\mathcal{V}_{M,\leq 8}$, which has 5512 elements, in a fair time[8]. We then compute

$$\begin{aligned}\mathcal{R}_0 &= (\mathcal{E} \cup -\mathcal{S}(L)) \cap \mathcal{V}_M = \mathcal{E} \cap \mathcal{V}_M \cap \mathbb{Q}_{\leq 8} = \mathcal{E} \cap \mathcal{V}_{M,\leq 8} \\ &= \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, 1, \frac{3}{2}, \frac{7}{4}, 2, \frac{5}{2}, 3, \frac{7}{2}, 4, 5, 6, 7, 8\right\}.\end{aligned}$$

Then, in order to compute $\mathcal{R}_1$, we first compute (the finite set)

$$\bigcup_{(\ell^\alpha,\beta) \in \mathcal{P}(L)} \ell^{-\alpha}(\psi(\mathcal{R}_0) - \beta)$$

and, then, we compute its intersection with $\mathcal{V}_{M,\leq 8}$. We obtain

$$\mathcal{R}_1 = \mathcal{R}_0 \cup \left\{-\frac{1}{16}, -\frac{1}{32}\right\}.$$

Iterating this process, we find $\mathcal{R}_2 = \mathcal{R}_1$. Thus, $\mathcal{R} = \mathcal{R}_1$ is a set with 21 elements.

The next step of Algorithm 61 consists in computing a basis of $\mathcal{C}_{\mathcal{R}}$. As explained in the proof of Theorem 62, this amounts to solve the linear system

$$F_\delta((f_\gamma)_{\gamma \in \mathcal{R}}) = 0, \quad \delta \in \psi(\mathcal{R})$$

with $\sharp\psi(\mathcal{R}) = \sharp\mathcal{R} = 21$ equations given by (83). We may gather these equations in a matrix whose columns are indexed by the elements of $\mathcal{R}$ and whose rows are indexed by the elements of $\psi(\mathcal{R})$. The coefficient of the entry (λ, γ) of this matrix is the coefficient of z^λ in

$$L(z^\gamma) = z^{1+4\gamma} + (z-1)z^{2\gamma} - 2z^\gamma.$$

To save space, we shall not reproduce here the square matrix of this system. After calculations, we find that the kernel of this matrix has dimension 1, generated by some explicit $(f_\gamma)_{\gamma \in \mathcal{R}}$. This means that $\mathcal{C}_{\mathcal{R}}$ has dimension 1 and is generated by $\sum_{\gamma \in \mathcal{R}} f_\gamma z^\gamma$.

Last, Algorithm 61 returns $\sum_{\gamma \in \mathcal{E}} f_\gamma z^\gamma$. Replacing the f_γ by their explicit values, we find (95).

[8]With a basic desktop computer and the computer algebra software Giac/Xcas it took us less than a minute.

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