A PURITY THEOREM FOR MAHLER EQUATIONS

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Abstract. The principal aim of this paper is to establish a purity theorem for Mahler functions that is reminiscent of famous purity theorems for G-functions by D. and G. Chudnovsky and for E-functions (and, more generally, for holonomic arithmetic Gevrey series) by Y. André. Our approach is based on a preliminary study of independent interest of the nature of the solutions of Mahler equations. Roughly speaking, we show that any Mahler equation admits a complete basis of solutions formed from what we call generalized Mahler series, which are sums involving Puiseux series, Hahn series of a very special type and solutions of inhomogeneous equations of order 1 with constant coefficients; such bases of solutions can be compared to those of differential equations given by Turrittin's theorem. In the light of B. Adamczewski, J. P. Bell and D. Smertnig's recent height gap theorem, we introduce a natural filtration on the set of generalized Mahler series according to the arithmetic growth of the coefficients of the Puiseux series involved in their decomposition. This filtration has 5 pieces. Our purity theorem states that the membership of a generalized Mahler series to one of the three largest pieces of this filtration propagates to any other generalized Mahler series solution of its minimal Mahler equation. We also show that this statement does not extend to the first two pieces.

Contents

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1. Introduction and statement of the main results

The primary aim of this article is to establish a purity theorem for Mahler functions that is reminiscent of famous purity theorems for G- and E-functions and, more generally, for holonomic arithmetic Gevrey series by D. and G. Chudnovsky and Y. André respectively. Although there is no concrete link between the latter results and those of the present article, they played a fundamental role in the genesis of our work, so we start by briefly recalling them.

Following Y. André in [And00], we say that a power series

$$f = \sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$$

with coefficients in the field of algebraic numbers $\overline{\mathbb{Q}}$ is an arithmetic Gevrey series of order $s \in \mathbb{Q}$ if there exists $C > 0$ such that

- for all $n \in \mathbb{Z}_{\geqslant 0}$, the maximum of the moduli of the galoisian conjuguates of $b_n := \frac{a_n}{n!^s}$ is bounded by C^{n+1};
- there exists a sequence of positive integers $(d_n)_{n\geqslant 0}$ such that, for all $n \in \mathbb{Z}_{\geqslant 0}$, $d_n \leqslant C^{n+1}$ and $d_n b_0, d_n b_1, \ldots, d_n b_n$ are algebraic integers.

An holonomic arithmetic Gevrey series of order 0 (resp. -1) is nothing but a G-function (resp. E-function) in the sense of C. L. Siegel [Sie29, Sie14]. By holonomic, we mean solution of a nonzero linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$.

To state the purity theorem for these series, it is convenient to introduce the differential $\mathbb{C}[z]$-algebra $\mathrm{NGA}\{z\}_s$ of arithmetic Nilsson-Gevrey series of order s. An element of $\mathrm{NGA}\{z\}_s$ is by definition a $\mathbb{C}$-linear combination of terms of the form

$$u(z)z^{\alpha}\log^j(z)$$

where $\alpha \in \mathbb{Q}$, $j \in \mathbb{Z}_{\geqslant 0}$, and $u(z)$ is an arithmetic Gevrey series of order s.

The following fondamental purity theorem is due to D. and G. Chudnovsky for $s = 0$ in [CC85] (the proof contains a slight mistake corrected by Y. André in [And89, Chapter VI]) and to Y. André for $s \neq 0$ in [And00] (see also [And03]).

Theorem 1 (Y. André, D. and G. Chudnovsky). *Let y be an arithmetic Gevrey series of order $s \in \mathbb{Q}$ or, more generally, an element of $\mathrm{NGA}\{z\}_s$ solution of a nonzero linear differential equation $\Psi y = 0$ with coefficients in $\mathbb{C}(z)$. We assume Ψ of minimal order μ. Then:*

(1) if $s \leqslant 0$, Ψ admits a full basis of solutions in $\mathrm{NGA}\{z\}_s$;

(2) if $s > 0$, Ψ admits a full basis of solutions of the form $e^{\alpha_i z^{-\frac{1}{s}}} y_i$ with $y_i \in \mathrm{NGA}\{z\}_s$ and $\alpha_i \in \overline{\mathbb{Q}}$.

Remark 2. *Any linear differential equation of order μ with coefficients in $\mathbb{C}(z)$ – or, more generally, in $\mathbb{C}(\{z\})$ – has a basis of solutions made of*

$\mathbb{C}$-linear combinations of terms of the form

$$u(z)z^{\alpha}\log^{j}(z)e^{Q(z^{-1/\mu!})}$$

where $\alpha \in \mathbb{C}$, $j \in \mathbb{Z}_{\geqslant 0}$, $u(z)$ is a Gevrey series and $Q(X) \in X\mathbb{C}[X]$. Theorem 1 shows in particular that the fact that the differential operator under consideration is a nonzero differential operator of minimal order annihilating an holonomic arithmetic Gevrey series imposes severe restrictions on the α, $u(z)$ and $Q(X)$ involved in its basis of solutions.

We now come to the subject of our study, the Mahler equations. Let $p \geqslant 2$ be an integer. By p-Mahler equation, we mean a linear functional equation of the form

$$a_0(z)f(z) + a_1(z)f(z^p) + \cdots + a_d(z)f(z^{p^d}) = 0 \tag{1}$$

with coefficients $a_0, \ldots, a_d$ in the field

$$\mathbb{K}_{\infty} = \overline{\mathbb{Q}}(z^{\frac{1}{*}}) = \bigcup_{k \in \mathbb{Z}_{\geqslant 1}} \overline{\mathbb{Q}}(z^{\frac{1}{k}})$$

of ramified rational functions such that $a_0 a_d \neq 0$. A solution of such an equation will be called a p-Mahler function. If such a solution is a power series (resp. a Puiseux series, a Hahn series, *etc.*), we will say that it is a p-Mahler power series (resp. a p-Mahler Puiseux series, a p-Mahler Hahn series, *etc.*).

Although the solutions of Mahler equations are very different in nature from those of differential equations, certain properties bring them close to arithmetic Gevrey series. Indeed, on the one hand, it is well-known that any p-Mahler series $f \in \overline{\mathbb{Q}}[[z]]$ is an arithmetic Gevrey series of order 0 [Dum93, Chap. 3, Cor. 8 and Th. 6] (but, be careful, if it is not rational, then f is not holonomic [Ran92, BCR13], and isn't even differentially algebraic over $\mathbb{C}(z)$ [ADH21]). On the other hand, the famous refinement of the Siegel-Shidlovskii theorem due to F. Beukers in [Beu06] and reproved by Y. André in [And14] admits a mahlerian analogue proved by P. Philippon [Phi15] and supplemented by B. Adamczewski and the first author in [AF17], which brings p-Mahler functions closer to E-functions. Note that another proof of Philippon's result, in the spirit of [And14], was subsequently given by L. Nagy and T. Szamuely in [NS20] and that a third proof was given by B. Adamczewski and the first author in [AF23]. It is properties like these that have encouraged us to investigate a possible extension of the above purity theorem to Mahler equations.

A natural motivation for looking at the growth properties of the coefficients of p-Mahler series or, more generally, of p-Mahler Hahn series comes from the Bombieri-Dwork conjecture predicting that the minimal differential equation of a G-function comes from geometry. In the light of this conjecture, it is natural to ask whether a p-Mahler Hahn series whose coefficients have a special growth has a special nature. For results in this direction, we refer to Section 1.3.2.

The theory of Mahler equations is a dynamic and fast-growing research area. Since the pioneering work of Mahler in [Mah29, Mah30a, Mah30b], numerous articles have been written on Mahler equations, which has known important recent developments; see for instance [Pel09, Ngu11, Ngu12, NN12, BCR13, BBC15, Phi15, BCZ16, AB17, AF17, AF18, CDDM18, DHR18, Roq18, Ada19, BCCD19, SS19, ADH21, Roq21, AZ22, FP22, MNS22, ABS23, AF23, AZ23, Pou23, AF24a, AF24b, FR24a] and the references therein, to name but a few recent articles. We believe that the results presented in the present paper will be useful for further work on Mahler equations.

The main results of the present paper are described in Sections 1.1 and 1.2 below. In Section 1.1, we outline our main results about the structure of the solutions of p-Mahler equations at 0. Theorem 4, which is the outcome of this study, shows that any p-Mahler equation has a full basis of solutions consisting of what we call generalized p-Mahler series. This result, of independent interest, is a necessary prerequisite for the statement and proof of our purity theorem, to which Section 1.2 is devoted. A number of comments, especially in connection with our forthcoming paper [FR24b], are given in Section 1.3.

1.1. Solving p-Mahler equations.

1.1.1. *Hahn series and p-Mahler equations.* Hahn series are a key ingredient for solving p-Mahler equations of the form (1). We let $\mathscr{H} = \overline{\mathbb{Q}}((z^{\mathbb{Q}}))$ be the field of Hahn series with coefficients in $\overline{\mathbb{Q}}$ and value group $\mathbb{Q}$. This field contains the field

$$\mathscr{P} = \bigcup_{k\in\mathbb{Z}_{\geqslant 1}} \overline{\mathbb{Q}}((z^{\frac{1}{*}}))$$

of Puiseux series as a subfield but it is much bigger. Roughly speaking, Hahn series are a generalization of Puiseux series allowing arbitrary exponents of the indeterminate as long as the set that supports them forms a well-ordered set; we refer to Section 2 for details. The interest of the Hahn series in our context lies in the following result: the difference field $(\mathscr{H}, \phi_p)$, where ϕ_p is the field automorphism of $\mathscr{H}$ sending $f(z)$ on $f(z^p)$, has a difference ring extension $(\mathscr{R}, \phi_p)$ with field of constants $\mathscr{R}^{\phi_p} = \{f \in \mathscr{R} \mid \phi_p(f) = f\}$ equal to $\overline{\mathbb{Q}}$ such that

- for any $c \in \overline{\mathbb{Q}}^{\times}$, there exists $e_c \in \mathscr{R}$ which is not a zero divisor satisfying $\phi_p(e_c) = ce_c$;
- there exists $\ell \in \mathscr{R}$ satisfying $\phi_p(\ell) = \ell + 1$;
- any p-Mahler equation of the form (1) has a full basis[1] of solutions $y_1, \ldots, y_d \in \mathscr{R}$ of the form

$$y_i = \sum_{(c,j)\in\overline{\mathbb{Q}}^{\times}\times\mathbb{Z}_{\geqslant 0}} f_{i,c,j} e_c \ell^j \tag{2}$$

[1]We say that a p-Mahler equation of order μ as "full basis" of solutions of a given form, if it has μ $\overline{\mathbb{Q}}$-linearly independent solutions of the given form.

where the sum is finite and the $f_{i,c,j}$ belong to $\mathscr{H}$.

We refer to Section 4.1 (and, in particular, to Proposition 25) for details and references.

1.1.2. *Generalized p-Mahler series and p-Mahler equations.* The first main result of this paper – namely Theorem 4 below – gives precise informations on the Hahn series $f_{i,c,j}$ involved in (2). It ensures that they have a very special form: they are linear combinations with coefficients in the ring of p-Mahler Puiseux series of specific Hahn series, denoted by $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$, that we shall now introduce.

For any $t \in \mathbb{Z}_{\geqslant 1}$, $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_t) \in \mathbb{Z}_{\geqslant 0}^t$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t) \in (\overline{\mathbb{Q}}^{\times})^t$ and $\boldsymbol{a} = (a_1, \ldots, a_t) \in \mathbb{Q}_{>0}^t$, we consider the Hahn series

$$\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z) = \sum_{k_1,\ldots,k_t \geqslant 1} k_1^{\alpha_1} \cdots k_t^{\alpha_t} \lambda_1^{k_1} \lambda_2^{k_1+k_2} \cdots \lambda_t^{k_1+\cdots+k_t} z^{-\frac{a_1}{p^{k_1}} - \frac{a_2}{p^{k_1+k_2}} - \cdots - \frac{a_t}{p^{k_1+k_2+\cdots+k_t}}} \in \mathscr{H}.$$

When $t = 0$, $\mathbb{Z}_{\geqslant 0}^t$, $(\overline{\mathbb{Q}}^{\times})^t$ and $\mathbb{Q}_{>0}^t$ have just one element, namely the empty vector () and, in this case, we write $\xi_{(),(),()}(z) = 1$. In what follows, we let

$$\boldsymbol{\Lambda} = \bigcup_{t \in \mathbb{Z}_{\geqslant 0}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^{\times})^t \times \mathbb{Q}_{>0}^t$$

be the set of possible values for the parameters $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a})$.

Definition 3. *A generalized p-Mahler series is an element of $\mathcal{R}$ of the form*

$$\sum_{(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}} f_{c,j} e_c \ell^j \tag{3}$$

where the sum has finite support and where the $f_{c,j}$ are Hahn series of the following form

$$f_{c,j} = \sum_{(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \boldsymbol{\Lambda}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \tag{4}$$

where the sum has finite support and where the $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \mathscr{P}$ are p-Mahler Puiseux series.

Theorem 4. *Any p-Mahler equation of the form (1) has a full basis of generalized p-Mahler series solutions, i.e., it has d $\overline{\mathbb{Q}}$-linearly independent generalized p-Mahler series solutions $y_1, \ldots, y_d \in \mathcal{R}$.*

In fact, we will obtain this result as a by-product of the construction of fundamental matrices of solutions of a very specific form of p-Mahler systems, which are reminiscent of the fundamental matrices of solutions of differential systems given by Turrittin's theorem; as this requires further notations, we say no more about this result of independent interest here and refer the reader to Section 4 and, especially, to Theorem 27.

Note that the decomposition (3) of a generalized p-Mahler series into a $\mathscr{H}$-linear combination of the $e_c\ell^j$ is unique, but that this is not the case for the decomposition (3)-(4) into a $\mathscr{P}$-linear combination of the $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j$. The following definition and proposition remedy this problem.

Definition 5. *We will say that the decomposition (3)-(4) is standard if the $\boldsymbol{a} = (a_1, \ldots, a_t)$ involved in the support of the sum in (4) have entries in the set $\mathbb{N}_{(p)}$ of positive rational numbers whose denominators are relatively prime with p and whose numerators are not divisible by p.*

Proposition 6. *Any generalized p-Mahler series has a unique standard decomposition.*

Remark 7. *We will prove (see Remark 22) that the Hahn series $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ are p-Mahler Hahn series. Since the property of being a solution of a p-Mahler equation is stable by sums and product, we obtain that any generalized p-Mahler series is a solution of a p-Mahler equation.*

1.2. **Purity Theorem.** Our purity theorem (Theorem 11; see also Theorem 14) involves growth conditions for generalized p-Mahler series inspired by the recent paper [ABS23] by B. Adamczewski, J. P. Bell and D. Smertnig. Let us briefly recall their main result.

1.2.1. *Growth of the coefficients of p-Mahler power series.* In [ABS23], B. Adamczewski, J. P. Bell and D. Smertnig study the asymptotic growth of the coefficients of p-Mahler power series with coefficients in $\overline{\mathbb{Q}}$, as measured by their logarithmic Weil height. Their main result is the following height gap theorem, which shows that there are five different growth behaviors.

Theorem 8 ([ABS23, Prop. 5.2])**.** *Any p-Mahler Puiseux series $f = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma \in \mathscr{P}$ satisfies one of the following mutually exclusive properties[2]:*

($\mathcal{O}\Omega_1$) $\mathit{h}(f_\gamma) \in \mathcal{O} \cap \Omega(H(\gamma))$;
($\mathcal{O}\Omega_2$) $\mathit{h}(f_\gamma) \in \mathcal{O} \cap \Omega(\log^2 H(\gamma))$;
($\mathcal{O}\Omega_3$) $\mathit{h}(f_\gamma) \in \mathcal{O} \cap \Omega(\log H(\gamma))$;
($\mathcal{O}\Omega_4$) $\mathit{h}(f_\gamma) \in \mathcal{O} \cap \Omega(\log\log H(\gamma))$;
($\mathcal{O}\Omega_5$) $\mathit{h}(f_\gamma) \in \mathcal{O}(1)$.

In this result and throughout this paper, $H(\alpha)$ denotes the Weil height of $\alpha \in \overline{\mathbb{Q}}$ and $\mathit{h}(\alpha) = \log H(\alpha)$ its logarithmic Weil height (see [Wal00] for details and references). Roughly speaking, they measure the "complexity" of the algebraic number α. For instance, when $\gamma = \frac{a}{b}$ is a rational number, with $a, b \in \mathbb{Z}$ relatively prime, $H(\gamma) = \max\{|a|, |b|\}$. Moreover, for any $(a_\gamma)_{\gamma\in\mathbb{Q}}, (b_\gamma)_{\gamma\in\mathbb{Q}} \in \mathbb{R}^{\mathbb{Q}}$, the notation $a_\gamma = \mathcal{O}(b_\gamma)$ means that there exists $C > 0$ such that, for all but finitely many $\gamma \in \mathbb{Q}$, we have $|a_\gamma| \leqslant C|b_\gamma|$ and the notation $a_\gamma = \Omega(b_\gamma)$ means that there exists $c > 0$ such that, for infinitely many $\gamma \in \mathbb{Q}$, we have $|a_\gamma| > c|b_\gamma|$.

[2]Strictly speaking, this result is only proved for power series and for p-Mahler equations with coefficients in $\overline{\mathbb{Q}}(z)$, but the extension to Puiseux series and to p-Mahler equations with coefficients in $\overline{\mathbb{Q}}(z^{\frac{1}{*}})$ is straightforward.

1.2.2. *Purity theorem.* Theorem 8 reveals five $\mathcal{O}$-growth conditions for Puiseux series: we say that $f = \sum_\gamma f_\gamma z^\gamma \in \mathscr{P}$ satisfies

- $(\mathcal{O}_1)$ if $\mathit{h}(f_\gamma) = \mathcal{O}(H(\gamma))$;
- $(\mathcal{O}_2)$ if $\mathit{h}(f_\gamma) = \mathcal{O}(\log^2 H(\gamma))$;
- $(\mathcal{O}_3)$ if $\mathit{h}(f_\gamma) = \mathcal{O}(\log H(\gamma))$;
- $(\mathcal{O}_4)$ if $\mathit{h}(f_\gamma) = \mathcal{O}(\log\log H(\gamma))$;
- $(\mathcal{O}_5)$ if $\mathit{h}(f_\gamma) = \mathcal{O}(1)$.

We extend these $\mathcal{O}$-growth conditions to generalized p-Mahler series as follows.

Definition 9. *We say that a generalized p-Mahler series f satisfies $(\mathscr{P}-\mathcal{O}_r)$ for some $r \in \{1,2,3,4,5\}$ if it admits a decomposition of the form (3)-(4) such that all the Puiseux series $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ satisfy $(\mathcal{O}_r)$.*

Proposition 10. *A generalized p-Mahler series f satisfies $(\mathscr{P}-\mathcal{O}_r)$ if and only if the Puiseux series $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ involved in its standard decomposition satisfy $(\mathcal{O}_r)$.*

It is clear that, for any $r \in \{1,\dots,4\}$, the condition $(\mathscr{P}-\mathcal{O}_{r+1})$ is stronger than $(\mathscr{P}-\mathcal{O}_r)$ in the sense that any generalized p-Mahler series satisfying $(\mathscr{P}-\mathcal{O}_{r+1})$ also satisfies $(\mathscr{P}-\mathcal{O}_r)$. Moreover, it follows from Theorem 8 that any generalized p-Mahler series satisfies $(\mathscr{P}-\mathcal{O}_1)$. Therefore, the five growth conditions $(\mathscr{P}-\mathcal{O}_1)$ to $(\mathscr{P}-\mathcal{O}_5)$ induce the following filtration on the set of generalized p-Mahler series:

$$\begin{aligned}
&\{\text{generalized } p\text{-Mahler series}\}\\
&\quad= \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{P}-\mathcal{O}_1)\}\\
&\qquad\supsetneq \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{P}-\mathcal{O}_2)\}\\
&\qquad\quad\supsetneq \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{P}-\mathcal{O}_3)\}\\
&\qquad\qquad\supsetneq \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{P}-\mathcal{O}_4)\}\\
&\qquad\qquad\quad\supsetneq \{\text{generalized } p\text{-Mahler series satisfying } (\mathscr{P}-\mathcal{O}_5)\}.
\end{aligned}$$

This filtration has 5 pieces. We are now ready to state our purity theorem guarantying that the membership of a generalized p-Mahler series to one of the three largest pieces of this filtration propagates to any other generalized Mahler series solution of its minimal Mahler equation.

Theorem 11 (Purity Theorem)**.** *Let f be a generalized p-Mahler series satisfying $(\mathscr{P}-\mathcal{O}_r)$ for some $r \in \{1,2,3\}$. Then, the minimal p-Mahler equation of f over $\mathbb{K}_\infty$ has a full basis of generalized p-Mahler series solutions satisfying $(\mathscr{P}-\mathcal{O}_r)$.*

Remark 12. *1) Considering the minimal p-Mahler equation is of course essential for the conclusions of Theorem 11 to hold. The constant function* 1*, which satisfies $(\mathscr{P}-\mathcal{O}_3)$, is solution of the equation*

$$(z - z^2 - 2z^3)y(z) + (-1 - z + z^2 + 2z^3 + 2z^4)y(z^2) + (1 - 2z^4)y(z^4) = 0.$$

Of course, this equation is not minimal with respect to 1*. The rational function* $\frac{1}{1-2z}$ *is also solution of this equation but it is does not satisfy* $(\mathscr{P}-\mathcal{O}_3)$*.*

2) Theorem 11 do not extend to $r \in \{4,5\}$*. See Section 7 for counterexamples.*

3) In view of Theorem 8, it would be natural to consider, for any $r \in \{1,2,3,4,5\}$*, the growth condition* $(\mathscr{P}-\mathcal{O}\Omega_r)$ *defined as follows: we say that a generalized p-Mahler series f satisfies* $(\mathscr{P}-\mathcal{O}\Omega_r)$ *if it satisfies* $(\mathscr{P}-\mathcal{O}_r)$ *and if at least one of the Puiseux series* $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ *involved in its standard decomposition satisfies* $(\mathcal{O}\Omega_r)$*. We emphasize that it is not possible to replace the condition* $(\mathscr{P}-\mathcal{O}_r)$ *by* $(\mathscr{P}-\mathcal{O}\Omega_r)$ *in Theorem 11. Indeed, the equation*

$$(1-2z)y(z)+(-1+2z-z^2+3z^3-3z^4)y(z^2)+(z^2-3z^3+3z^4)y(z^4)=0$$

is the minimal 2*-Mahler equation associated with some Laurent series*

$$-z^{-1}+3z+6z^2+6z^3+21z^4+21z^5+60z^6+99z^7+234z^8+408z^9+870z^{10}+\mathcal{O}(z^{11})$$

satisfying $(\mathscr{P}-\mathcal{O}\Omega_1)$*. Another solution is the constant function* 1*, which obviously does not satisfy* $(\mathscr{P}-\mathcal{O}\Omega_1)$*.*

1.3. **Comments in connection with [FR24b].** In this Section, we gather remarks related to our forthcoming paper [FR24b] where we will study the growth of the coefficients of p-Mahler Hahn series.

1.3.1. *Purity theorem in terms of Hahn series.* Instead of considering the generalized p-Mahler series as $\mathscr{P}$-linear combinations of the $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j$ as we did in Definition 9, we can see them as $\mathscr{H}$-linear combinations of the $e_c \ell^j$ as in (3). This point of view leads to the following alternative extension of the growth conditions $(\mathcal{O}_1)$ to $(\mathcal{O}_5)$ to generalized p-Mahler series and to an alternative purity theorem.

Definition 13. *We say that a generalized p-Mahler series f satisfies* $(\mathscr{H}-\mathcal{O}_r)$ *for some* $r \in \{1,2,3,4,5\}$ *if the Hahn series* $f_{c,j}$ *involved in the decomposition* (3) *satisfy (the obvious extension to Hahn series of)* $(\mathcal{O}_r)$*.*

The corresponding purity theorem reads as follows.

Theorem 14 (Purity Theorem, Hahn series version)**.** *Let f be a generalized p-Mahler series satisfying* $(\mathscr{H}-\mathcal{O}_r)$ *for some* $r \in \{1,2,3\}$*. Then, the minimal p-Mahler equation of f over* $\mathbb{K}_\infty$ *has a full basis of generalized p-Mahler series solutions satisfying* $(\mathscr{H}-\mathcal{O}_r)$*.*

This result will be proved in [FR24b]; actually, we will prove that Theorem 11 and Theorem 14 are equivalent, *i.e.*, that conditions $(\mathscr{P}-\mathcal{O}_r)$ and $(\mathscr{H}-\mathcal{O}_r)$ are equivalent for $r \in \{1,2,3\}$.

1.3.2. *Regularity, automaticity and growth.* In [FR24b], inspired by the work of K. S. Kedlaya in [Ked17], we introduce notions of quasi-p-regular and quasi-p-automatic Hahn series. These notions are extensions to Hahn series of the classical notions of p-regular and p-automatic series [AS03]. In [FR24b], we characterize the quasi-p-regular and quasi-p-automatic Hahn series among the p-Mahler Hahn series in terms of the growth of their coefficients, generalizing results from [ABS23] relative to p-Mahler power series; we prove that, for any p-Mahler Hahn series f, we have:

- f is quasi-p-regular if and only if f satisfies $(\mathscr{H} - \mathcal{O}_3)$;
- f is quasi-p-automatic if and only if f satisfies $(\mathscr{H} - \mathcal{O}_5)$.

Therefore, the case $r = 3$ of Theorem 14 implies:

Corollary 15. *The minimal p-Mahler equation of a quasi-p-regular Hahn series has a full basis of solutions made of $\overline{\mathbb{Q}}$-linear combinations of terms of the form $f e_c \ell^j$ where f is a quasi-p-regular Hahn series, $c \in \overline{\mathbb{Q}}^{\times}$ and $j \in \mathbb{Z}_{\geqslant 0}$.*

1.3.3. *Growth of the coefficients of p-Mahler Hahn series.* In [FR24b], we establish a height gap theorem extending Theorem 8 to p-Mahler Hahn series.

1.4. **Notations.** In this Section, we list the main notations used in this paper.

We let $\mathbb{Z}$ be the ring of relative integers, $\mathbb{Q}$ be the field of rational numbers, $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. We let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

Given a subset E of $\mathbb{R}$ and $\delta \in \mathbb{R}$, we let $E_{\geqslant \delta}$ denote the set of elements of E greater than or equal to δ. The sets $E_{>\delta}$, $E_{\leqslant \delta}$ and $E_{<\delta}$ are defined in a similar way.

Given a ring R, we let $R^{\times}$ denote the multiplicative group of units of R.

We let $\mathbb{N}_{(p)}$ denote the set of positive rational numbers whose denominators are relatively prime with p and whose numerators are not divisible by p. Note that, for any positive rational number $\gamma \in \mathbb{Q}_{>0}$, there exists an unique integer $k \in \mathbb{Z}$ such that $p^k \gamma \in \mathbb{N}_{(p)}$.

We set

$$\mathbf{\Lambda} = \bigcup_{t \in \mathbb{Z}_{\geqslant 0}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^{\times})^t \times \mathbb{Q}_{>0}^t$$

and

$$\mathbf{\Lambda}_{\mathrm{st}} = \bigcup_{t \in \mathbb{Z}_{\geqslant 0}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^{\times})^t \times \mathbb{N}_{(p)}^t. \tag{5}$$

We let $\overline{\mathbb{Q}}[[z]]$ be the ring of power series with coefficients in $\overline{\mathbb{Q}}$. We let $\overline{\mathbb{Q}}((z))$ be the fraction field of $\overline{\mathbb{Q}}[[z]]$, that is, the field of Laurent series over $\overline{\mathbb{Q}}$. We let $\mathbb{K}_{\infty} = \bigcup_{k \in \mathbb{Z}_{\geqslant 1}} \overline{\mathbb{Q}}(z^{\frac{1}{k}})$ denote the field of ramified rational functions with algebraic coefficients, $\mathscr{P}$ denote the field of Puiseux series over $\overline{\mathbb{Q}}$ and $\mathscr{H}$ denote the field of Hahn series over $\overline{\mathbb{Q}}$ and value group $\mathbb{Q}$. We have the following tower of fields:

$$\overline{\mathbb{Q}}(z) \subset \mathbb{K}_{\infty} \subset \mathscr{P} \subset \mathscr{H}.$$

For any $F = \sum_{\gamma \in \mathbb{Q}} F_{\gamma} z^{\gamma} \in \mathrm{M}_d(\mathscr{H})$, we set

$$F^0 = F_0, \qquad F^{<0} = \sum_{\gamma \in \mathbb{Q}_{<0}} F_{\gamma} z^{\gamma}, \qquad F^{>0} = \sum_{\gamma \in \mathbb{Q}_{>0}} F_{\gamma} z^{\gamma}, \tag{6}$$

so that

$$F = F^{<0} + F^{0} + F^{>0}.$$

The notation $\mathscr{R}$ refers to a difference ring extension of $\mathscr{H}$ described in Section 4.1

1.5. **Organization of the paper.** In Section 2, we recall the definition of the field of Hahn series. In Section 3, we recall the dictionary between Mahler equations, systems and modules. Section 4 is devoted to the construction of a fundamental matrix of solutions at 0 of a given p-Mahler system. Our main result with this respect is Theorem 27. It is the cornerstone of the present paper. Theorem 4 is deduced from Theorem 27 at the end of Section 4. In Section 5, we establish the properties of the standard decomposition for generalized p-Mahler series stated in the introduction, namely Propositions 6 and 10. One may skip this Section at first reading. Section 6 is devoted to the proof of our purity theorem, Theorem 11. In Section 7, we give an example showing that our purity theorem cannot be extended to conditions $(\mathscr{P} - \mathbb{O}_4)$ or $(\mathscr{P} - \mathbb{O}_5)$.

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2. Hahn series

2.1. **Definitions.** We denote by

$$\mathscr{H} = \overline{\mathbb{Q}}((z^{\mathbb{Q}}))$$

the field of Hahn series over $\overline{\mathbb{Q}}$ and with value group $\mathbb{Q}$. An element of $\mathscr{H}$ is an $(f_\gamma)_{\gamma\in\mathbb{Q}} \in \overline{\mathbb{Q}}^{\mathbb{Q}}$ whose support

$$\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}}) = \{\gamma \in \mathbb{Q} \mid f_\gamma \neq 0\}$$

is well-ordered (*i.e.*, any nonempty subset of $\operatorname{supp}(f)$ has a least element) with respect to the restriction to $\operatorname{supp}((f_\gamma)_{\gamma\in\mathbb{Q}})$ of the usual order on $\mathbb{Q}$. An element $(f_\gamma)_{\gamma\in\mathbb{Q}}$ of $\mathscr{H}$ is usually (and will be) denoted by

$$f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma.$$

The sum and product of two elements $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma$ and $g = \sum_{\gamma\in\mathbb{Q}} g_\gamma z^\gamma$ of $\mathscr{H}$ are given by

$$f + g = \sum_{\gamma\in\mathbb{Q}} (f_\gamma + g_\gamma) z^\gamma$$

and

$$fg = \sum_{\gamma\in\mathbb{Q}} \left(\sum_{\gamma'+\gamma''=\gamma} f_{\gamma'} g_{\gamma''} \right) z^\gamma.$$

(The fact that the supports of f and g are well-ordered implies that there are only finitely many $(\gamma',\gamma'')\in\mathbb{Q}\times\mathbb{Q}$ such that $\gamma'+\gamma''=\gamma$ and $f_{\gamma'}g_{\gamma''}\neq 0$, so the sums $\sum_{\gamma'+\gamma''=\gamma} f_{\gamma'}g_{\gamma''}$ are meaningful.)

The field $\mathscr{H}$ of Hahn series contains the field $\mathscr{P}$ of Puiseux series as a subfield but it is much bigger. A typical example of Hahn series which is not a Puiseux series is given by

$$\xi_{0,1,1}=\sum_{k\geqslant 0} z^{-\frac{1}{p^k}}.$$

We let $\mathscr{H}^{<0}$ be the set made of the $f\in\mathscr{H}$ such that $\operatorname{supp}(f)\subset\mathbb{Q}_{<0}$.

We say that a family $(f_i)_{i\in I}$ of elements of $\mathscr{H}$ is summable if the following properties are satisfied :

- the set $\bigcup_{i\in I}\operatorname{supp}(f_i)$ is well-ordered;
- for any $\gamma\in\mathbb{Q}$, the set
$$\{i\in I \mid \gamma\in\operatorname{supp}(f_i)\}$$
is finite.

In this case, we define

$$\sum_{i\in I} f_i=\sum_{i\in I}\left(\sum_{\gamma\in\mathbb{Q}} f_{i,\gamma}\right) z^{\gamma}\in\mathscr{H}$$

where $f_i=\sum_{\gamma\in\mathbb{Q}} f_{i,\gamma}z^{\gamma}$. We have the following elementary lemma; see [Roq24, Lemma 31].

Lemma 16. *For any $f\in\mathscr{H}^{<0}$, the family $(\phi_p^k(f))_{k\leqslant -1}$ of elements of $\mathscr{H}$ is summable.*

2.2. **The Hahn series $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ and the $\overline{\mathbb{Q}}$-vector spaces $\mathcal{V}_s$.** We shall now focus our attention on the Hahn series $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ introduced in Section 1.1.2 and on some related $\overline{\mathbb{Q}}$-vector spaces that will play an essential role in the present paper.

For any $t\in\mathbb{Z}_{\geqslant 1}$, $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_t)\in\mathbb{Z}_{\geqslant 0}^t$, $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_t)\in(\overline{\mathbb{Q}}^{\times})^t$ and $\boldsymbol{a}=(a_1,\ldots,a_t)\in\mathbb{Q}_{>0}^t$, we consider the Hahn series

$$\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)=\sum_{k_1,\ldots,k_t\geqslant 1} k_1^{\alpha_1}\cdots k_t^{\alpha_t}\lambda_1^{k_1}\lambda_2^{k_1+k_2}\cdots\lambda_t^{k_1+\cdots+k_t} z^{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_t}{p^{k_1+k_2+\cdots+k_t}}}\in\mathscr{H}.$$

To prove that this definition is legitimate and indeed gives a Hahn series, we have to prove that, for any $\gamma\in\mathbb{Q}$, there are at most finitely many $(k_1,\ldots,k_t)\in\mathbb{Z}_{\geqslant 1}^t$ such that $\gamma=-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_t}{p^{k_1+k_2+\cdots+k_t}}$ and that the support of $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)$ is well-ordered; this follows from the following lemma.

Lemma 17. *For any $t\in\mathbb{Z}_{\geqslant 1}$ and $\boldsymbol{a}=(a_1,\ldots,a_t)\in\mathbb{Q}_{>0}^t$, we have :*

- *the set*
$$\left\{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_t}{p^{k_1+k_2+\cdots+k_t}} \mid (k_1,\ldots,k_t)\in\mathbb{Z}_{\geqslant 1}^t\right\}$$
is well-ordered;
- *for any $\gamma\in\mathbb{Q}$, there are at most finitely many $(k_1,\ldots,k_t)\in\mathbb{Z}_{\geqslant 1}^t$ such that*
$$\gamma=-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_t}{p^{k_1+k_2+\cdots+k_t}}.$$

Proof. We argue by induction on $t\in\mathbb{Z}_{\geqslant 1}$.

Base case $t=1$. Lemma 17 is obvious for $t=1$.

Inductive step $t\to t+1$. We assume that Lemma 17 holds true for some $t\in\mathbb{Z}_{\geqslant 1}$. We want to prove that, for any $\boldsymbol{a}=(a_1,\ldots,a_{t+1})\in\mathbb{Q}_{>0}^{t+1}$,

(i) the set
$$\left\{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_{t+1}}{p^{k_1+k_2+\cdots+k_{t+1}}} \mid (k_1,\ldots,k_{t+1})\in\mathbb{Z}_{\geqslant 1}^{t+1}\right\}$$
is well-ordered;

(ii) for any $\gamma\in\mathbb{Q}$, there are at most finitely many $(k_1,\ldots,k_{t+1})\in\mathbb{Z}_{\geqslant 1}^{t+1}$ such that
$$\gamma=-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_{t+1}}{p^{k_1+k_2+\cdots+k_{t+1}}}.$$

By induction hypothesis (applied to $(a_2,\ldots,a_{t+1})\in\mathbb{Q}_{>0}^t$),
$$f=\sum_{k_2,\ldots,k_{t+1}\geqslant 1} z^{-a_1-\frac{a_2}{p^{k_2}}-\cdots-\frac{a_{t+1}}{p^{k_2+\cdots+k_{t+1}}}}$$
is a well-defined element of $\mathscr{H}^{<0}$. Lemma 16 guaranties that $(\phi_p^{k_1}(f))_{k_1\leqslant -1}$ is summable; this means exactly that :

(iii) the set
$$\bigcup_{k_1\in\mathbb{Z}_{\geqslant 1}}\operatorname{supp}(\phi_p^{k_1}(f))=$$
$$\left\{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_{t+1}}{p^{k_1+k_2+\cdots+k_{t+1}}} \mid (k_1,\ldots,k_{t+1})\in\mathbb{Z}_{\geqslant 1}^{t+1}\right\}$$
is well-ordered;

(iv) for any $\gamma\in\mathbb{Q}$, there are at most finitely many $k_1\in\mathbb{Z}_{\geqslant 1}$ such that
$$\gamma\in\operatorname{supp}(\phi_p^{k_1}(f))=$$
$$\left\{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_{t+1}}{p^{k_1+k_2+\cdots+k_{t+1}}} \mid (k_2,\ldots,k_{t+1})\in\mathbb{Z}_{\geqslant 1}^{t}\right\}.$$

Now, (iii) ensures that (i) is true. Moreover, (iv) combined with the fact that, by induction, for any $k_1\in\mathbb{Z}_{\geqslant 1}$, there are finitely many $(k_2,\ldots,k_{t+1})\in\mathbb{Z}_{\geqslant 1}^t$

such that $\gamma = -\frac{a_1}{p^{k_1}} - \frac{a_2}{p^{k_1+k_2}} - \cdots - \frac{a_{t+1}}{p^{k_1+k_2+\cdots+k_{t+1}}}$, implies that (ii) is true. This concludes the induction. □

We extend the notation $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ to the case $t=0$ as follows: $\mathbb{Z}^0_{\geqslant 0} = (\overline{\mathbb{Q}}^\times)^0 = \mathbb{Q}^0_{>0} = \{()\}$ is the set with one element $()$ and we set

$$\xi_{(),(),()}(z) = 1.$$

For any $s \in \mathbb{Z}_{\geqslant 0}$, we consider the following $\overline{\mathbb{Q}}$-vector space

$$\mathcal{V}_s = \mathrm{Span}_{\overline{\mathbb{Q}}}\Big(\{z^{-\gamma}\xi_{(),(),()} \mid \gamma \in \mathbb{Q}_{>0}\} \cup \bigcup_{t\in\{1,\ldots,s\}} \{z^{-\gamma}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \mid \gamma \in \mathbb{Q}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \mathbb{Z}^t_{\geqslant 0} \times (\overline{\mathbb{Q}}^\times)^t \times \mathbb{Q}^t_{>0}\}\Big).$$

Of course, $(\mathcal{V}_s)_{s\in\mathbb{Z}_{\geqslant 0}}$ is a non decreasing sequence of $\overline{\mathbb{Q}}$-vector spaces; its limit is denoted by

$$\begin{aligned} \mathcal{V} &= \bigcup_{s\in\mathbb{Z}_{\geqslant 0}} \mathcal{V}_s \\ &= \mathrm{Span}_{\overline{\mathbb{Q}}}\Big(\{z^{-\gamma}\xi_{(),(),()} \mid \gamma \in \mathbb{Q}_{>0}\} \\ &\quad \cup \bigcup_{t\in\mathbb{Z}_{\geqslant 1}} \{z^{-\gamma}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \mid \gamma \in \mathbb{Q}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \mathbb{Z}^t_{\geqslant 0} \times (\overline{\mathbb{Q}}^\times)^t \times \mathbb{Q}^t_{>0}\}\Big). \end{aligned} \tag{7}$$

We will now state and prove a few technical lemmas about the Hahn series $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ and the $\overline{\mathbb{Q}}$-vector spaces $\mathcal{V}_s$ that will be used later in the paper. The reader can ignore them on first reading and return to them when they are used in later demonstrations.

Lemma 18. *The map $\phi_p : \mathscr{H} \to \mathscr{H}$ induces a $\overline{\mathbb{Q}}$-linear automorphism of $\mathcal{V}_s$.*

Proof. Follows straightforwardly from the equality

$$\phi_p(\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)) = \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p\boldsymbol{a}}(z).$$

□

Lemma 19. *For any $c \in \overline{\mathbb{Q}}^\times$ and $\alpha \in \mathbb{Z}_{\geqslant 0}$, the $\overline{\mathbb{Q}}$-linear map*

$$\begin{aligned} \mathscr{H}^{<0} &\to \mathscr{H}^{<0} \\ h &\mapsto \sum_{k\leqslant -1} k^\alpha c^k \phi_p^k(h) \end{aligned}$$

is well-defined and sends $\mathcal{V}_s$ in $\mathcal{V}_{s+1}$.

Proof. The fact that this map is well-defined follows immediately from Lemma 16. Its $\overline{\mathbb{Q}}$-linearity is obvious. It remains to prove that this map sends $\mathcal{V}_s$ in $\mathcal{V}_{s+1}$. By $\overline{\mathbb{Q}}$-linearity, it is sufficient to consider the case $h = z^{-\gamma}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ with $\boldsymbol{\alpha} = (\alpha_1,\ldots,\alpha_t) \in \mathbb{Z}^t_{\geqslant 0}$, $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_t) \in (\overline{\mathbb{Q}}^\times)^t$ and $\boldsymbol{a} = (a_1,\ldots,a_t) \in \mathbb{Q}^t_{>0}$

for some $t \in \{0, \ldots, s\}$ and $\gamma \in \mathbb{Q}_{\geqslant 0}$ (resp. $\gamma \in \mathbb{Q}_{>0}$) if $t \geqslant 1$ (resp. $t = 0$). If $t = 0$, we have

$$\sum_{k \leqslant -1} k^{\alpha} c^{k} \phi_p^k(z^{-\gamma}) = \sum_{k \geqslant 1} (-k)^{\alpha} c^{-k} z^{-\frac{\gamma}{p^k}} = (-1)^{\alpha} \xi_{(\alpha),(c^{-1}),(\gamma)}(z) \in \mathcal{V}_1 \subset \mathcal{V}_{s+1}$$

as claimed. If $t \geqslant 1$, we have

$$\begin{aligned}
&\sum_{k \leqslant -1} k^{\alpha} c^{k} \phi_p^k(z^{-\gamma} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)) \\
&= \sum_{k_0 \geqslant 1} (-k_0)^{\alpha} c^{-k_0} z^{-\frac{\gamma}{p^{k_0}}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{\frac{1}{p^{k_0}}}) \\
&= (-1)^{\alpha} \sum_{k_0 \geqslant 1} \sum_{k_1,\ldots,k_t \geqslant 1} k_0^{\alpha} k_1^{\alpha_1} \cdots k_t^{\alpha_t} c^{-k_0} \lambda_1^{k_1} \cdots \lambda_t^{k_1+\cdots+k_t} \\
&\qquad\qquad \times z^{-\frac{\gamma}{p^{k_0}} - \frac{a_1}{p^{k_0+k_1}} - \frac{a_2}{p^{k_0+k_1+k_2}} - \cdots - \frac{a_t}{p^{k_0+k_1+k_2+\cdots+k_t}}} \\
&= (-1)^{\alpha} \sum_{k_0,k_1,\ldots,k_t \geqslant 1} k_0^{\alpha} k_1^{\alpha_1} \cdots k_t^{\alpha_t} \left(\frac{1}{c\lambda_1 \cdots \lambda_t}\right)^{k_0} \lambda_1^{k_0+k_1} \cdots \lambda_t^{k_0+k_1+\cdots+k_t} \\
&\qquad\qquad \times z^{-\frac{\gamma}{p^{k_0}} - \frac{a_1}{p^{k_0+k_1}} - \frac{a_2}{p^{k_0+k_1+k_2}} - \cdots - \frac{a_t}{p^{k_0+k_1+k_2+\cdots+k_t}}} \\
&= (-1)^{\alpha} \xi_{\boldsymbol{\beta},\boldsymbol{\tau},\boldsymbol{b}} \in \mathcal{V}_{s+1}
\end{aligned}$$

with $\boldsymbol{\beta} = (\alpha, \alpha_1, \ldots, \alpha_t)$, $\boldsymbol{\tau} = \left(\frac{1}{c\lambda_1 \cdots \lambda_t}, \lambda_1, \ldots, \lambda_t\right)$ and $\boldsymbol{b} = (\gamma, a_1, \ldots, a_t)$. □

Lemma 20. *For any $s, s' \in \mathbb{Z}_{\geqslant 0}$ and any $(h(z), h'(z)) \in \mathcal{V}_s \times \mathcal{V}_{s'}$, we have $h(z)h'(z) \in \mathcal{V}_{s+s'}$.*

Proof. We first note that

$$(8) \quad \mathcal{V}_s = \mathrm{Span}_{\overline{\mathbb{Q}}} \Big(\{ z^{-\gamma} \widetilde{\xi}_{(),(),()} \mid \gamma \in \mathbb{Q}_{>0} \} \\ \cup \bigcup_{t \in \{1,\ldots,s\}} \{ z^{-\gamma} \widetilde{\xi}_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \mid \gamma \in \mathbb{Q}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^{\times})^t \times \mathbb{Q}_{>0}^t \} \Big)$$

where

$$\widetilde{\xi}_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z) = \sum_{1 \leqslant k_1 < \cdots < k_t} k_1^{\alpha_1} \cdots k_t^{\alpha_t} \lambda_1^{k_1} \lambda_2^{k_2} \cdots \lambda_t^{k_t} z^{-\frac{a_1}{p^{k_1}} - \frac{a_2}{p^{k_2}} - \cdots - \frac{a_t}{p^{k_t}}} \in \mathscr{H}$$

with the convention $\widetilde{\xi}_{(),(),()}(z) = 1$. Indeed, (8) follows immediately from the following facts:

- any $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ is a $\overline{\mathbb{Q}}$-linear combinaison of certain $\widetilde{\xi}_{\boldsymbol{\alpha}',\boldsymbol{\lambda},\boldsymbol{a}}$ with $\boldsymbol{\alpha}' \in \mathbb{Z}^t_{\geqslant 0}$ as the following formula clearly shows:

$$\begin{aligned}(9)\quad & \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z) = \\ & \sum_{k_1,\dots,k_t \geqslant 1} k_1^{\alpha_1}\cdots k_t^{\alpha_t}\lambda_1^{k_1}\lambda_2^{k_1+k_2}\cdots\lambda_t^{k_1+\cdots+k_t} z^{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}-\cdots-\frac{a_t}{p^{k_1+k_2+\cdots+k_t}}} \\ & = \sum_{1\leqslant l_1<\cdots<l_t} l_1^{\alpha_1}(l_2-l_1)^{\alpha_2}\cdots(l_t-l_{t-1})^{\alpha_t}\lambda_1^{l_1}\lambda_2^{l_2}\cdots\lambda_t^{l_t} z^{-\frac{a_1}{p^{l_1}}-\frac{a_2}{p^{l_2}}-\cdots-\frac{a_t}{p^{l_t}}};\end{aligned}$$

- any $\widetilde{\xi}_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ is a $\overline{\mathbb{Q}}$-linear combinaison of certain $\xi_{\boldsymbol{\alpha}',\boldsymbol{\lambda},\boldsymbol{a}}$ with $\boldsymbol{\alpha}' \in \mathbb{Z}^t_{\geqslant 0}$ as the following formula clearly shows:

$$\begin{aligned}(10)\quad \widetilde{\xi}_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z) &= \sum_{1\leqslant k_1<\cdots<k_t} k_1^{\alpha_1}\cdots k_t^{\alpha_t}\lambda_1^{k_1}\lambda_2^{k_2}\cdots\lambda_t^{k_t} z^{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_2}}-\cdots-\frac{a_t}{p^{k_t}}} \\ &= \sum_{l_1,\dots,l_t\geqslant 1} l_1^{\alpha_1}(l_1+l_2)^{\alpha_2}\cdots(l_1+\cdots+l_t)^{\alpha_t}\lambda_1^{l_1}\lambda_2^{l_1+l_2}\cdots\lambda_t^{l_1+\cdots+l_t} \\ &\qquad \times z^{-\frac{a_1}{p^{l_1}}-\frac{a_2}{p^{l_1+l_2}}-\cdots-\frac{a_t}{p^{l_1+\cdots+l_t}}}.\end{aligned}$$

Thus, it is sufficient to prove that, for any $t,t' \in \mathbb{Z}_{\geqslant 1}$, for any $\boldsymbol{\alpha} \in \mathbb{Z}^t_{\geqslant 0}$, $\boldsymbol{\lambda} \in \overline{\mathbb{Q}}^t$ and $\boldsymbol{a} \in \mathbb{Q}^t_{>0}$, for any $\boldsymbol{\alpha}' \in \mathbb{Z}^{t'}_{\geqslant 0}$, $\boldsymbol{\lambda}' \in \overline{\mathbb{Q}}^{t'}$ and $\boldsymbol{a}' \in \mathbb{Q}^{t'}_{>0}$, we have $\widetilde{\xi}_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)\widetilde{\xi}_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}(z) \in \mathcal{V}_{t+t'}$. We prove this in the case $t = t' = 2$; the general case is similar but requires unpleasant notations. We have

$$\begin{aligned}(11)\quad & \widetilde{\xi}_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)\widetilde{\xi}_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}(z) \\ & \qquad = \sum_{\substack{1\leqslant k_1<k_2\\ 1\leqslant k_1'<k_2'}} k_1^{\alpha_1}k_2^{\alpha_2}{k'}_1^{\alpha_1'}{k'}_2^{\alpha_2'}\lambda_1^{k_1}\lambda_t^{k_2}{\lambda'}_1^{k_1'}{\lambda'}_2^{k_2'} z^{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_2}}-\frac{a_1'}{p^{k_1'}}-\frac{a_2'}{p^{k_2'}}}.\end{aligned}$$

But, this sum can be decomposed as follows:

$$\begin{aligned}(12)\quad \sum_{\substack{1\leqslant k_1<k_2\\ 1\leqslant k_1'<k_2'}} &= \sum_{1\leqslant k_1<k_2<k_1'<k_2'} + \sum_{1\leqslant k_1<k_1'<k_2<k_2'} + \sum_{1\leqslant k_1<k_1'<k_2'<k_2} \\ (13)\quad & + \sum_{1\leqslant k_1'<k_1<k_2<k_2'} + \sum_{1\leqslant k_1'<k_1<k_2'<k_2} + \sum_{1\leqslant k_1'<k_2'<k_1<k_2} \\ (14)\quad & + \sum_{1\leqslant k_1=k_1'<k_2<k_2'} + \sum_{1\leqslant k_1=k_1'<k_2'<k_2} + \sum_{1\leqslant k_1'<k_1=k_2'<k_2} \\ (15)\quad & + \sum_{1\leqslant k_1<k_2=k_1'<k_2'} + \sum_{1\leqslant k_1<k_1'<k_2=k_2'} + \sum_{1\leqslant k_1'<k_1<k_2=k_2'} \\ (16)\quad & + \sum_{1\leqslant k_1'=k_1<k_2=k_2'}\end{aligned}$$

Each sum in (the right hand side of) (12) and (13) is equal to $\widetilde{\xi}_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{b}}$ for some $\boldsymbol{\beta} \in \mathbb{Z}^4_{\geqslant 0}$, $\boldsymbol{\mu} \in \overline{\mathbb{Q}}^4$ and $\boldsymbol{b} \in \mathbb{Q}^4_{>0}$. Each sum in (14) and (15) is equal to $\widetilde{\xi}_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{b}}$ for some $\boldsymbol{\beta} \in \mathbb{Z}^3_{\geqslant 0}$, $\boldsymbol{\mu} \in \overline{\mathbb{Q}}^3$ and $\boldsymbol{b} \in \mathbb{Q}^3_{>0}$. The sum in (16) is equal to $\widetilde{\xi}_{\boldsymbol{\beta},\boldsymbol{\mu},\boldsymbol{b}}$ for some $\boldsymbol{\beta} \in \mathbb{Z}^2_{\geqslant 0}$, $\boldsymbol{\mu} \in \overline{\mathbb{Q}}^2$ and $\boldsymbol{b} \in \mathbb{Q}^2_{>0}$. Hence, (11) is in $\mathcal{V}_4$, as expected. □

Lemma 21. *For every $j \in \mathbb{Z}$, $\boldsymbol{\alpha} \in \mathbb{Z}^s_{\geqslant 0}$, $\boldsymbol{\lambda} \in \overline{\mathbb{Q}}^s$ and $\boldsymbol{a} \in \mathbb{Q}^s_{>0}$, we have*

$$(17)\quad \begin{aligned} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{p^j}) &= (\lambda_1\cdots\lambda_s)^j \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z) \\ &+ \sum_{\boldsymbol{\alpha}'\in\{0,\ldots,\alpha_1-1\}\times\mathbb{Z}^{s-1}_{\geqslant 0},\boldsymbol{\lambda}'\in\overline{\mathbb{Q}}^s,\boldsymbol{a}'\in\mathbb{Q}^s_{>0}} p_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}(z)\xi_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}(z) \\ &+ \sum_{(\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}')\in\bigcup_{t\in\{0,\ldots,s-1\}}\mathbb{Z}^t_{\geqslant 0}\times\overline{\mathbb{Q}}^t\times\mathbb{Q}^t_{>0}} p_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}(z)\xi_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}(z) \end{aligned}$$

for some $p_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}(z) \in \mathscr{P}^{<0} = \overline{\mathbb{Q}}[z^{-\frac{1}{*}}]^{<0}$. Moreover, if $\boldsymbol{a} \in \mathbb{Z}^s_{>0}$ and $j \in \mathbb{Z}_{\geqslant 1}$, then we have a decomposition of the form (17) such that the $p_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}(z)$ belong to $\overline{\mathbb{Q}}[z^{-1}]^{<0}$ and such that the $\boldsymbol{a}'$ involved in the support of the sums in (17) have entries in $\mathbb{Z}_{>0}$.

Proof. Let us first prove the result for $j = 1$. Let $\lambda_0 = \lambda_1\cdots\lambda_s$. We have

$$\begin{aligned} &\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^p) \\ =\ &\lambda_0 \sum_{k_1\geqslant 0,k_2,\ldots,k_s\geqslant 1} (k_1+1)^{\alpha_1}k_2^{\alpha_2}\cdots k_s^{\alpha_s}\lambda_1^{k_1}\cdots\lambda_s^{k_1+\cdots+k_s} z^{-\frac{a_1}{p^{k_1}}-\cdots-\frac{a_s}{p^{k_1+k_2+\cdots+k_s}}} \\ =\ &\lambda_0 \sum_{i=0}^{\alpha_1}\binom{\alpha_1}{i}\sum_{k_1,\ldots,k_s\geqslant 1} k_1^i k_2^{\alpha_2}\cdots k_s^{\alpha_s}\lambda_1^{k_1}\cdots\lambda_s^{k_1+\cdots+k_s} z^{-\frac{a_1}{p^{k_1}}-\cdots-\frac{a_s}{p^{k_1+k_2+\cdots+k_s}}} \\ &+\lambda_0 \sum_{k_2,\ldots,k_s\geqslant 1} k_2^{\alpha_2}\cdots k_s^{\alpha_s}\lambda_2^{k_2}\cdots\lambda_s^{k_2+\cdots+k_s} z^{-a_1-\frac{a_2}{p^{k_2}}-\cdots-\frac{a_s}{p^{k_2+\cdots+k_s}}} \end{aligned}$$

and, hence,

$$\begin{aligned} &\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^p) - \lambda_1\cdots\lambda_s\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z) \\ &\qquad = \lambda_0\sum_{i=0}^{\alpha_1-1}\binom{\alpha_1}{i}\xi_{(i,\alpha_2,\ldots,\alpha_s),\boldsymbol{\lambda},\boldsymbol{a}} + \lambda_0 z^{-a_1}\xi_{\boldsymbol{\beta},\boldsymbol{\tau},\boldsymbol{b}} \end{aligned}$$

where $\boldsymbol{\beta} = (\alpha_2,\ldots,\alpha_s)$, $\boldsymbol{\tau} = (\lambda_2,\ldots,\lambda_s)$ and $\boldsymbol{b} = (a_2,\ldots,a_s)$. The latter expression has the desired form.

The case of an arbitrary $j \in \mathbb{Z}_{\geqslant 1}$ follows from an easy induction using the particular case $j = 1$.

We now consider the case $j=-1$. We have

$$\begin{aligned}
&\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{p^{-1}}) \\
=\ &\lambda_0^{-1}\sum_{k_1\geqslant 2,k_2,\dots,k_s\geqslant 1}(k_1-1)^{\alpha_1}k_2^{\alpha_2}\cdots k_s^{\alpha_s}\lambda_1^{k_1}\cdots\lambda_s^{k_1+\cdots+k_s}z^{-\frac{a_1}{p^{k_1}}-\cdots-\frac{a_s}{p^{k_1+k_2+\cdots+k_s}}} \\
=\ &\sum_{i=0}^{\alpha_1}\gamma_i\sum_{k_1,\dots,k_t\geqslant 1}k_1^i k_2^{\alpha_2}\cdots k_s^{\alpha_s}\lambda_1^{k_1}\cdots\lambda_s^{k_1+\cdots+k_s}z^{-\frac{a_1}{p^{k_1}}-\cdots-\frac{a_s}{p^{k_1+k_2+\cdots+k_s}}} \\
&-\sum_{i=0}^{\alpha_1}\gamma_i\lambda_0\sum_{k_2,\dots,k_t\geqslant 1}k_2^{\alpha_2}\cdots k_s^{\alpha_s}\lambda_2^{k_2}\cdots\lambda_s^{k_2+\cdots+k_s}z^{-\frac{a_1}{p}-\frac{a_2}{p^{k_2+1}}-\cdots-\frac{a_s}{p^{k_2+\cdots+k_s+1}}}
\end{aligned}$$

where $\gamma_i=\lambda_0^{-1}(-1)^{i-\alpha_1}\binom{\alpha_1}{i}$. Hence

$$\begin{aligned}
&\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{p^{-1}})-(\lambda_1\cdots\lambda_s)^{-1}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z) \\
&\qquad=\sum_{i=0}^{\alpha_1-1}\gamma_i\xi_{(i,\alpha_2,\dots,\alpha_s),\boldsymbol{\lambda},\boldsymbol{a}}-\sum_{i=0}^{\alpha_1}\gamma_i\lambda_0 z^{-\frac{a_1}{p}}\xi_{\boldsymbol{\beta},\boldsymbol{\tau},\boldsymbol{b}}
\end{aligned}$$

where $\boldsymbol{\beta}=(\alpha_2,\dots,\alpha_s)$, $\boldsymbol{\tau}=(\lambda_2,\dots,\lambda_s)$ and $\boldsymbol{b}=(a_2/p,\dots,a_s/p)$. The latter expression has the desired form.

The case of an arbitrary $j\in\mathbb{Z}_{\leqslant -1}$ follows from an easy induction using the particular case $j=-1$. □

Remark 22. *It is easily seen that, if f is a p-Mahler Hahn series and if g is a Hahn series such that $g-a\phi_p(g)=f$ for some $a\in\mathbb{K}_\infty$, then g is a p-Mahler Hahn series as well. Using this fact and the case $j=1$ of Lemma 21, one can prove by induction on s and on α_1 (with the notations of loc. cit.) that the $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ are p-Mahler Hahn series.*

3. Equations, systems and modules

We recall that we let

$$\mathbb{K}_\infty=\overline{\mathbb{Q}}(z^{\frac{1}{*}})=\bigcup_{k\in\mathbb{Z}_{\geqslant 1}}\overline{\mathbb{Q}}(z^{\frac{1}{k}})$$

denote the field of ramified rational functions with coefficients in $\overline{\mathbb{Q}}$. We consider the field automorphism

$$\begin{array}{rcl}\phi_p:\mathbb{K}_\infty & \to & \mathbb{K}_\infty\\ f(z) & \mapsto & f(z^p).\end{array}$$

The pair $(\mathbb{K}_\infty,\phi_p)$ is a difference field. We let (K,ψ) be a difference field extension of $(\mathbb{K}_\infty,\phi_p)$, *i.e.*, K is a field extension of $\mathbb{K}_\infty$ and ψ is a field automorphism of K extending ϕ_p. Here are some examples:

- $K=\mathbb{K}_\infty$;
- $K=\mathscr{P}$ the field of Puiseux series over $\overline{\mathbb{Q}}$;
- $K=\mathscr{H}$ the field of Hahn series over $\overline{\mathbb{Q}}$ with value group $\mathbb{Q}$;

endowed with the natural extension of ϕ_p still denoted by ϕ_p and given, for any $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in K$, by

$$\phi_p(f) = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^{p\gamma}.$$

In what follows, we will often drop ϕ_p out of our notations, *e.g.*, we will simply speak of the difference field extension K of $\mathbb{K}_\infty$ instead of the difference field extension (K, ψ) of $(\mathbb{K}_\infty, \phi_p)$.

3.1. **Equations.** By p-Mahler equation over K, we mean an equation of the form

$$a_0 f + a_1\phi_p(f) + \cdots + a_d\phi_p^d(f) = 0 \tag{18}$$

with $a_0, \ldots, a_d \in K$ and $a_0 a_d \neq 0$.

3.2. **Systems.** By p-Mahler system over K, we mean a system of the form

$$\phi_p(F) = AF$$

with $A \in \mathrm{GL}_d(K)$.

We say that two systems $\phi_p(F) = AF$ and $\phi_p(F) = BF$ with $A, B \in \mathrm{GL}_d(K)$ are K-equivalent if there exists $F \in \mathrm{GL}_d(K)$ such that $\phi_p(F)A = BF$. Such an F is called a gauge transformation.

3.3. **From equations to systems.** Sometimes we will start with an equation, but it will be more convenient to work with a system. We recall that any p-Mahler equation can be converted into a p-Mahler system as follows: the equation (18) is equivalent to the system

$$\phi_p(F) = AF \tag{19}$$

where

$$F = \begin{pmatrix} f \\ \phi_p(f) \\ \vdots \\ \phi_p^d(f) \end{pmatrix} \text{ and } A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\frac{a_0}{a_d} & -\frac{a_1}{a_d} & \cdots & \cdots & -\frac{a_{d-1}}{a_d} \end{pmatrix}.$$

3.4. **Modules.** We denote by

$$\mathcal{D}_K = K\langle \phi_p, \phi_p^{-1}\rangle$$

the Ore algebra of noncommutative Laurent polynomials with coefficients in K such that, for all $f \in K$,

$$\phi_p f = \phi_p(f)\phi_p.$$

A left $\mathcal{D}_K$-module of finite length will be called a p-Mahler module (over K). Note that a left $\mathcal{D}_K$-module has finite length if and only if the K-vector space obtained by restriction of scalars has finite dimension; by definition, the rank of a p-Mahler module is its dimension as a K-vector space.

Given two p-Mahler modules M and N, the notation $M \cong N$ means that M and N are isomorphic as left $\mathcal{D}_K$-modules.

3.5. **From systems to modules and** ***vice-versa*****.** It is sometimes useful to work with p-Mahler modules instead of p-Mahler systems and *vice-versa*. Let us briefly recall the correspondence between p-Mahler systems and p-Mahler modules.

One can associate to any p-Mahler system

$$\phi_p(Y) = AY \tag{20}$$

with $A \in \mathrm{GL}_d(K)$ a p-Mahler module M_A as follows. We consider the map $\Phi_A : K^d \to K^d$ defined by

$$\Phi_A(m) = A\phi_p(m)$$

(here ϕ_p acts component-wise on the elements of K^d seen as column vectors). The p-Mahler module M_A is then defined as follows: the underlying abelian group is K^d (its elements being seen as column vectors) and the action of $L = \sum a_i\phi_p^i \in \mathcal{D}_K$ on $m \in M_A$ is given by

$$Lm = \left(\sum a_i\phi_p^i\right) m = \sum a_i\Phi_A^i(m).$$

Conversely, we can attach to any p-Mahler module M, a p-Mahler system *via* the choice of a K-basis $\mathcal{B} = (e_1, \ldots, e_d)$ of M: the p-Mahler system associated with M, with respect to $\mathcal{B}$, is $\phi_p(Y) = AY$ where $A \in \mathrm{GL}_d(K)$ represents the action of ϕ_p on $\mathcal{B}$ (*i.e.*, the jth column of A represents $\phi_p(e_j)$ in the basis $\mathcal{B}$). We have $M \cong M_A$.

It is easily seen that two p-Mahler systems $\phi_p(Y) = AY$ and $\phi_p(Y) = BY$ with $A, B \in \mathrm{GL}_d(K)$ are K-equivalent, *i.e.*, that there exists $F \in \mathrm{GL}_d(K)$ such that $\phi_p(F)A = BF$, if and only if the corresponding p-Mahler modules M_A and M_B are isomorphic.

Last, we recall the following classical result, known as the cyclic vector lemma, ensuring that any Mahler module "comes form" an equation.

Proposition 23. *For any p-Mahler module M, there exists $L \in \mathcal{D}_K$ such that $M \cong \mathcal{D}_K/\mathcal{D}_K L$.*

For a proof, see for instance [HS99, Theorem B.2].

4. Fundamental matrices of solutions of Mahler systems and proof of Theorem 4

In this Section, we first show that any p-Mahler system

$$\phi_p(Y) = AY \tag{21}$$

with $A \in \mathrm{GL}_d(\mathscr{P})$ admits a fundamental matrix of solutions of the form

$$Y_0 = Fe_C$$

where F is an invertible $d \times d$ matrix with coefficients in $\mathscr{H}$ and where e_C is an invertible $d \times d$ matrix with coefficients in a certain difference ring extension $\mathcal{R}$ of $\mathscr{H}$ satisfying

$$\phi_p(e_C) = Ce_C$$

for some $C \in \mathrm{GL}_d(\overline{\mathbb{Q}})$. The principal aim of the remainder of this Section is to study in greater detail the nature of F. Our main results with this respect, namely Theorem 27 and Theorem 29, provide us with a decomposition of the form $F = F_1F_2$ where $F_1 \in \mathrm{GL}_d(\mathscr{P})$ and where $F_2 \in \mathrm{GL}_d(\mathcal{V})$ satisfy certain nice properties. These results will be of great importance for the proofs of the main results of the present paper.

The difference ring extension $\mathcal{R}$ of $\mathscr{H}$ and the matrix e_C alluded to above are built in Section 4.1. Theorems 27 and 29 are stated in Section 4.2. The remainder of Section 4 is devoted to the proofs of these results and, eventually, to the proof of Theorem 4.

4.1. A fundamental matrix of solutions of the form $Y_0 = Fe_C$.

Our first objective is to construct, for any $C \in \mathrm{GL}_d(\overline{\mathbb{Q}})$, a matrix e_C satisfying $\phi_p(e_C) = Ce_C$. In order to do so, we first need to introduce a certain difference ring extension $\mathcal{R}$ of $\mathscr{H}$.

4.1.1. *The difference ring $\mathcal{R}$.* In what follows, we let $\mathcal{R}$ be a difference ring extension of $\mathscr{H}$ with field of constants $\overline{\mathbb{Q}}$ such that:

- there exists $\ell \in \mathcal{R}$ satisfying $\phi_p(\ell) = \ell + 1$;
- for any $c \in \overline{\mathbb{Q}}^{\times}$, there exists $e_c \in \mathcal{R}$, which is not a zero divisor, satisfying $\phi_p(e_c) = ce_c$.

Such a ring $\mathcal{R}$ exists. Indeed, let $(X_c)_{c \in \overline{\mathbb{Q}}^{\times}}$ and Y be indeterminates over $\mathscr{H}$, and consider the quotient ring

$$\mathcal{R} := \mathscr{H}[(X_c)_{c \in \overline{\mathbb{Q}}^{\times}}, Y]/I$$

of the polynomial ring $\mathscr{H}[(X_c)_{c \in \overline{\mathbb{Q}}^{\times}}, Y]$ by its ideal I generated by $\{X_cX_d - X_{cd} \mid c, d \in \overline{\mathbb{Q}}^{\times}\} \cup \{X_1 - 1\}$. Let e_c (resp. ℓ) be the image of X_c (resp. Y) in $\mathcal{R}$, so that

$$\mathcal{R} = \mathscr{H}[(e_c)_{c \in \overline{\mathbb{Q}}^{\times}}, \ell].$$

We endow $\mathcal{R}$ with its unique ring automorphism ϕ_p extending $\phi_p : \mathscr{H} \to \mathscr{H}$ and such that

$$\forall c \in \overline{\mathbb{Q}}^{\times},\ \phi_p(e_c) = ce_c \text{ and } \phi_p(\ell) = \ell + 1.$$

Then, $(\mathcal{R}, \phi_p)$ is a difference ring extension of $(\mathscr{H}, \phi_p)$ with field of constants $\overline{\mathbb{Q}}$. We omit the proof of this assertion as it is entirely similar to the proof of the second assertion of [Roq18, Theorem 35].

4.1.2. *Definition of* e_C. We are now ready to construct a matrix e_C satisfying $\phi_p(e_C) = Ce_C$.

Lemma 24. *For any* $C \in \mathrm{GL}_d(\overline{\mathbb{Q}})$, *there exists* $e_C \in \mathrm{M}_d(\mathcal{R})$ *such that*

$$\phi_p(e_C) = Ce_C, \tag{22}$$

whose determinant $\det e_C$ *is not a zero divisor of* $\mathcal{R}$ *and whose entries are* $\overline{\mathbb{Q}}$*-linear combinaisons of elements of the form* $e_c\ell^j$ *with* $c \in \mathrm{Spec}(C)$ *and* $j \in \{0, \ldots, d-1\}$.

Proof. Let $C = UD$ be the multiplicative Dunford-Jordan decomposition of the matrix C:

- $D \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ is semisimple,
- $U \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ is unipotent,
- and $UD = DU$.

We recall that D and U belong to $\overline{\mathbb{Q}}[C]$. We set

$$\ell^{[k]} = \begin{cases} \binom{\ell}{k} = \frac{\ell(\ell-1)\cdots(\ell-k+1)}{k!} & \text{if } k \in \mathbb{Z}_{\geqslant 0}, \\ 0 & \text{if } k \in \mathbb{Z}_{\leqslant -1}. \end{cases}$$

It follows from the equality $\phi_p(\ell) = \ell + 1$ that

$$\phi_p(\ell^{[k]}) = \ell^{[k]} + \ell^{[k-1]}$$

and that (the finite sum)

$$e_U = \sum_{k \geqslant 0} \ell^{[k]}(U - I_d)^k \in \mathrm{M}_d(\mathcal{R})$$

satisfies

$$\phi_p(e_U) = Ue_U.$$

Note that e_U belongs to $\mathcal{R}[U] \subset \mathcal{R}[C]$ and that $\det e_U = 1$.

Moreover, we consider $P \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ and $c_1, \ldots, c_d \in \overline{\mathbb{Q}}^{\times}$ such that

$$D = P \operatorname{diag}(c_1, \ldots, c_d) P^{-1}$$

and we set

$$e_D = P \operatorname{diag}(e_{c_1}, \ldots, e_{c_n}) P^{-1} \in \mathrm{M}_d(\mathcal{R}).$$

This e_D is independent of P and satisfies

$$\phi(e_D) = De_D.$$

Moreover, $\det e_D = e_{c_1} \cdots e_{c_d}$ is not a zero divisor.

Since $DU = UD$, the matrices e_U and D commute and it follows from what precedes that

$$e_C = e_U e_D$$

has the required properties. □

4.1.3. *Existence of* $Y_0 = Fe_C$. According to [Roq21, Theorem 2], there exist $C \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ and $F \in \mathrm{GL}_d(\mathscr{H})$ such that

$$AF = \phi_p(F)C. \tag{23}$$

Considering the matrix e_C given by Lemma 24 and combining (22) and (23), we obtain that

$$Y_0 = Fe_C \in \mathrm{M}_d(\mathcal{R})$$

satisfies

$$\phi_p(Y_0) = AY_0$$

and that $\det Y_0 = \det F \det e_C$ is not a zero divisor of $\mathcal{R}$.

Note the following consequence in terms of p-Mahler equations.

Proposition 25. *Any p-Mahler equation of the form (1) has d $\overline{\mathbb{Q}}$-linearly independent solutions $y_1, \dots, y_d$ of the form (2).*

Proof. Indeed, the system $\phi_p(Y) = AY$ associated to this equation (as in Section 3.3) has a fundamental matrix of solutions of the form $Y_0 = Fe_C \in \mathrm{M}_d(\mathcal{R})$. The d elements of the first line of Y_0 are $\overline{\mathbb{Q}}$-linearly independent solutions of equation (1) of the expected form. □

The aim of what follows is to obtain more information on F. This is achieved with Theorem 27 and Theorem 29 below. In order to state with exactness these results, we first need to introduce some notation.

4.2. **Nature of the coefficients of F.** For any $s \in \mathbb{Z}_{\geqslant 1}$ and $\boldsymbol{r} = (r_1, \dots, r_s) \in \mathbb{Z}_{\geqslant 1}^s$ such that

$$r_1 + \cdots + r_s = d,$$

we let $\mathfrak{V}_{\boldsymbol{r}}$ be the set of matrices of the form

$$F = \begin{pmatrix} I_{r_1} & & & & & & \\ & \ddots & & & & & \\ & & I_{r_i} & & F_{i,j} & & \\ & 0 & & \ddots & & & \\ & & & & I_{r_j} & & \\ & & & & & \ddots & \\ & & & & & & I_{r_s} \end{pmatrix} \in \mathrm{GL}_d(\mathscr{H}) \tag{24}$$

such that

$$F_{i,j} \in \mathrm{M}_{r_i, r_j}(\mathcal{V}_{j-i}).$$

Lemma 26. $\mathfrak{V}_{\boldsymbol{r}}$ *is a subgroup of* $\mathrm{GL}_d(\mathscr{H})$.

Proof. Of course, $\mathfrak{V}_{\boldsymbol{r}}$ contains I_d. The fact that $\mathfrak{V}_{\boldsymbol{r}}$ is invariant by product follows straightforwardly from Lemma 20 and from the fact that the sets $\mathcal{V}_{j-i}$ are $\overline{\mathbb{Q}}$-vector spaces. It remains to prove that $\mathfrak{V}_{\boldsymbol{r}}$ is invariant by inversion.

Consider $F \in \mathfrak{V}_{\boldsymbol{r}}$. The (additive) Dunford-Jordan decomposition of F is given by $F = I_d + N$ where

$$N = \begin{pmatrix} 0_{r_1} & & & & & & \\ & \ddots & & & & & \\ & & 0_{r_i} & & F_{i,j} & & \\ & 0 & & \ddots & & & \\ & & & & 0_{r_j} & & \\ & & & & & \ddots & \\ & & & & & & 0_{r_s} \end{pmatrix}.$$

Thus, $F^{-1} = \sum_{k=0}^{d} (-1)^k N^k$. But, it follows from Lemma 20 that

$$N^k = \begin{pmatrix} 0_{r_1} & & & & & & \\ & \ddots & & & & & \\ & & 0_{r_i} & & N_{k;i,j} & & \\ & 0 & & \ddots & & & \\ & & & & 0_{r_j} & & \\ & & & & & \ddots & \\ & & & & & & 0_{r_s} \end{pmatrix}$$

with $N_{k;i,j} \in \mathrm{M}_{r_i,r_j}(\mathcal{V}_{j-i})$. Whence the result. □

The rest of Section 4 is devoted to the proof of the following results.

Theorem 27. *The Mahler system (21) has a fundamental matrix of solutions of the form*

$$F_1 F_2 e_C$$

where

- $F_1 \in \mathrm{GL}_d(\mathscr{P})$;
- $F_2 \in \mathfrak{V}_{\boldsymbol{r}}$ *for some* $s \in \mathbb{Z}_{\geqslant 1}$ *and* $\boldsymbol{r} = (r_1, \ldots, r_s) \in \mathbb{Z}_{\geqslant 1}^s$ *such that* $r_1 + \cdots + r_s = d$;
- $C \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ *is a block upper triangular matrix of the form*

$$C = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_s \end{pmatrix}$$

for some $(A_1, \ldots, A_s) \in \mathrm{GL}_{r_1}(\overline{\mathbb{Q}}) \times \cdots \times \mathrm{GL}_{r_s}(\overline{\mathbb{Q}})$.

Moreover, we can choose F_1 *and* F_2 *in such a way that*

$$F_1(z^p)\Theta(z) = A(z)F_1(z) \quad \textit{and} \quad F_2(z^p)C = \Theta(z)F_2(z)$$

for some block upper triangular matrix of the form

$$\Theta = \begin{pmatrix} A_1 & & \star \\ & \ddots & \\ 0 & & A_s \end{pmatrix} \tag{25}$$

with coefficients in $\bigcup_{\gamma \in \mathbb{Q}_{>0}} \overline{\mathbb{Q}}[z^{-\gamma}]$ *and constant term* C.

Remark 28. *One can always choose* $s = d$ *and* $r_1 = \cdots = r_d = 1$ *in Theorem 27 (follows from Theorem 27 itself by triangularizing the matrices* A_i*). However, we have stated Theorem 27 in this form for the following reason: if* L *is a Mahler operator associated to the system (21) by the cyclic vector lemma, then the proof of Theorem 27 shows that we can take for* $r_1, \ldots, r_s$ *the multiplicities of the slopes of* L*. This more precise information will not be exploited in this paper but could be of interest for other purposes.*

Theorem 29. *Assume that (21) has a fundamental matrix of solutions of the form* $F' e_{C'}$ *with* $F' \in \mathrm{GL}_d(\mathscr{H})$ *and* $C' \in \mathrm{GL}_d(\overline{\mathbb{Q}})$*. Let* F_1, F_2, C, Θ *be given by Theorem 27. Then, there exists a matrix* $R \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ *such that*

- $C' = R^{-1} C R$,
- $F' = F_1 F_2'$, *with* $F_2' = F_2 R \in \mathrm{GL}_d(\mathscr{V}_{d-1})$
- $F_2'(z^p) C' = \Theta(z) F_2'(z)$

The proof of Theorem 27 is given in Section 4.6. It rests on intermediate results given in the next three Sections. The proof of Theorem 29 is given in Section 4.7.

4.3. **First step of the proof of Theorem 27: triangularization by blocks.** In this section and in the rest of the paper, we will use the following notation: for any $A, F \in \mathrm{GL}_d(\mathscr{H})$, we set

$$F[A] := \phi_p(F) A F^{-1}.$$

The first step of our proof of Theorem 27 consists in proving the following result.

Proposition 30. *Consider a Mahler system*

$$\phi_p(Y) = AY \tag{26}$$

with $A \in \mathrm{GL}_d(\mathscr{P})$*. There exist* $s \in \mathbb{Z}_{\geqslant 1}$, $\boldsymbol{r} = (r_1, \ldots, r_s) \in \mathbb{Z}_{\geqslant 1}^s$ *such that* $r_1 + \cdots + r_s = d$, $(A_1, \ldots, A_s) \in \mathrm{GL}_{r_1}(\overline{\mathbb{Q}}) \times \cdots \times \mathrm{GL}_{r_s}(\overline{\mathbb{Q}})$ *and* $F \in \mathrm{GL}_d(\mathscr{P})$ *such that*

$$F[A] = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_s \end{pmatrix}. \tag{27}$$

Remark 31. *One can always choose* $s = d$ *and* $r_1 = \cdots = r_d = 1$*. However, we have stated Theorem 30 in this form for the following reason: if* L *is a Mahler operator associated to the system (26), then the proof of Proposition 30 shows that we can take for* $r_1, \ldots, r_s$ *the multiplicities of the slopes of* L*.*

The proof of Proposition 30 is given in Section 4.3.3 below. Actually, we will not prove this result directly but a reformulation of it in terms of p-Mahler modules given in Proposition 32.

4.3.1. *Reformulation of Proposition 30 in terms of p-Mahler modules.* Proposition 30 can be reformulated in terms of p-Mahler modules as follows:

Proposition 32. *Let M be a p-Mahler module over $\mathscr{P}$ of rank $d \geqslant 1$. There exist $s \in \mathbb{Z}_{\geqslant 1}$ and $\boldsymbol{r} = (r_1, \ldots, r_s) \in \mathbb{Z}_{\geqslant 1}^s$ such that $r_1 + \cdots + r_s = d$ and a filtration*

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

by p-Mahler sub-modules of M such that, for all $i \in \{0, \ldots, s-1\}$,

$$M_{i+1}/M_i \cong M_{A_i}$$

for some $A_i \in \mathrm{GL}_{r_i}(\overline{\mathbb{Q}})$.

Let us explain why this result is equivalent to Proposition 30.

Let us first assume that Proposition 30 is true. Let M be a p-Mahler module over $\mathscr{P}$ of rank $d \geqslant 1$. As recalled in (and with the notations of) Section 3, there exists $A \in \mathrm{GL}_d(\mathscr{P})$ such that $M \cong M_A$. By Proposition 30, $M \cong M_B$ for some block upper triangular matrix $B \in \mathrm{GL}_d(\mathscr{P})$ of the form (27). Of course, the existence of a filtration of M as in Proposition 32 is equivalent to the existence of a similar filtration for M_B and it is clear that M_B has such a filtration: if $(e_1, \ldots, e_d)$ is the canonical basis of $\mathscr{P}^d$ then

$$\{0\} = N_0 \subset N_1 = \oplus_{k=1}^{r_1} \mathscr{P} e_k \subset N_2 = \oplus_{k=1}^{r_1+r_2} \mathscr{P} e_k \subset \cdots \\ \cdots \subset N_s = \oplus_{k=1}^{r_1+r_2+\cdots+r_s} \mathscr{P} e_k = M_B$$

is a filtration by submodules of M_B such that

$$N_{i+1}/N_i \cong M_{A_i}$$

for all $i \in \{0, \ldots, s-1\}$. This shows that Proposition 30 implies Proposition 32.

Conversely, assume that Proposition 32 is true. Let $A \in \mathrm{GL}_d(\mathscr{P})$ and consider the p-Mahler module M_A. By Proposition 32, there exists a filtration

$$\{0\} = N_0 \subset N_1 \subset \cdots \subset N_s = M_A$$

by submodules of M_A of rank r_i such that, for all $i \in \{0, \ldots, s-1\}$,

$$N_{i+1}/N_i \cong M_{A_i}$$

for some $A_i \in \mathrm{GL}_{r_i}(\overline{\mathbb{Q}})$. Let $\mathcal{B} = (e_1, \ldots, e_d)$ be a basis of M such that, for all $i \in \{1, \ldots, s\}$, $(e_1, \ldots, e_{r_1+\cdots+r_i})$ is a basis of N_i and such that the action of ϕ_p on N_{i+1}/N_i is represented in the basis $e_{r_i+1} + N_i, \ldots, e_{r_{i+1}} + N_i$ by A_i. Then, the p-Mahler system $\phi_p Y = BY$ associated to M with respect to the basis $\mathcal{B}$ (see Section 3) has the form (27). Since the p-Mahler systems $\phi_p Y = AY$ and $\phi_p Y = BY$ are $\mathscr{P}$-equivalent, this yields the desired result. This shows that Proposition 32 implies Proposition 30.

This concludes the proof of the equivalence between Propositions 30 and 32.

We will prove Proposition 32 (and, hence, Proposition 30) in Section 4.3.3. This proof will rest on a factorization property of p-Mahler operators given in the next section.

4.3.2. *Factorization of p-Mahler operators.* Consider a p-Mahler operator

$$L = a_d\phi_p^d + a_{d-1}\phi_p^{d-1} + \cdots + a_0 \in \mathcal{D}_{\mathscr{P}} \tag{28}$$

with coefficients $a_0, \ldots, a_d \in \mathscr{P}$ such that $a_0 a_d \neq 0$. We let

- $\mu_1 < \cdots < \mu_s$ be the slopes of L with respective multiplicities $r_1, \ldots, r_s$;
- $c_{i,1}, \ldots, c_{i,r_i} \in \overline{\mathbb{Q}}^\times$ be the exponents (repeated with multiplicities) of L attached to the slope μ_i.

See [Roq24, Section 4] for these notions. For any $f = \sum_{\gamma\in\mathbb{Q}} f_\gamma z^\gamma \in \mathscr{H}$, we let $\operatorname{val}_z f = \min \operatorname{supp} f \in \mathbb{Q} \cup \{+\infty\}$ denote the valuation of f and, if $f \neq 0$, $\operatorname{cld}_z f = f_{\operatorname{val}_z f} \in \overline{\mathbb{Q}}\backslash\{0\}$ denote the coefficient of least degree.

Proposition 33. *The operator L has a factorization of the form*

$$L = aL_s \cdots L_1$$

where

- $a \in \mathscr{P}^\times$ *is such that* $\operatorname{val}_z a = \operatorname{val}_z a_0$*;*
- $\operatorname{cld}_z a = \prod_{i=1}^k \prod_{j=1}^{r_i} (-c_{i,j})^{-1} \operatorname{cld}_z a_0$*;*
- *the L_i are given by*

  $$L_i = (z^{\nu_i}\phi_p - c_{i,r_i})h_{i,r_i}^{-1} \cdots (z^{\nu_i}\phi_p - c_{i,1})h_{i,1}^{-1}$$

  for some $h_{i,j} \in \mathscr{P}^\times$ with $\operatorname{val}_z h_{i,j} = 0$, $\operatorname{cld}_z h_{i,j} = 1$ and

  $$\nu_i = (p-1)(p^{r_1+\cdots+r_{i-1}}(\mu_i - \mu_{i-1}) + \cdots + p^{r_1}(\mu_2 - \mu_1) + \mu_1).$$

Proof. This result is proved in [Roq24, Proposition 15] over $\mathscr{H}$ instead of $\mathscr{P}$; the proof in the present case is entirely similar. □

4.3.3. *Proof of Proposition 32.* According to the cyclic vector lemma (Proposition 23), there exists $L \in \mathcal{D}_{\mathscr{P}}$ such that $M \cong \mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}}L$. The factorization

$$L = aL_s \cdots L_1$$

given by Proposition 33 induces a filtration

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

by p-Mahler sub-modules of M such that, for all $i \in \{0, \ldots, s-1\}$, $M_{i+1}/M_i \cong \mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}}L_i$. But,

$$\mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}}L_i \cong \mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}}\widetilde{L}_i$$

with

$$\widetilde{L}_i = z^{\frac{\nu_i}{p-1}} L_i z^{-\frac{\nu_i}{p-1}} = (\phi_p - c_{i,r_i}) h_{i,r_i}^{-1} \cdots (\phi_p - c_{i,1}) h_{i,1}^{-1} = \sum_{j=0}^{r_i} b_j \phi_p^j$$

for some $b_j \in \mathscr{P}$ of non-negative z-adic valuation and such that $b_0(0) b_{r_i}(0) \neq 0$. It follows from [Roq18, Proposition 34] that the (system associated to the) module $\mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}} L_i \cong \mathcal{D}_{\mathscr{P}}/\mathcal{D}_{\mathscr{P}} \widetilde{L}_i$ is regular singular at 0 in the sense of [Roq18, Definition 33] and, hence, is isomorphic to M_{A_i} for some $A_i \in \mathrm{GL}_{r_i}(\overline{\mathbb{Q}})$. This concludes the proof of Proposition 32.

4.4. **Second step of the proof of Theorem 27: elimination of the part of positive valuation.** Consider $s \in \mathbb{Z}_{\geqslant 1}$ and $\boldsymbol{r} = (r_1, \ldots, r_s) \in \mathbb{Z}_{\geqslant 1}^s$ such that $r_1 + \cdots + r_s = d$.

We let $\mathfrak{H}_{\boldsymbol{r}}$ be the group of matrices of the form

$$F = \begin{pmatrix} I_{r_1} & & & & & & \\ & \ddots & & & & & \\ & & I_{r_i} & & F_{i,j} & & \\ & 0 & & \ddots & & & \\ & & & & I_{r_j} & & \\ & & & & & \ddots & \\ & & & & & & I_{r_s} \end{pmatrix} \in \mathrm{GL}_d(\mathscr{P}) \tag{29}$$

where $F_{i,j} \in \mathrm{M}_{r_i, r_j}(\mathscr{P})$.

We let $\mathscr{P}^{>0}$ be the subring of $\mathscr{P}$ made of the Puiseux series with support in $\mathbb{Q}_{>0}$. We let $\mathscr{P}^{\leqslant 0}$ be the subring of $\mathscr{P}$ made of the Puiseux series with support in $\mathbb{Q}_{\leqslant 0}$; thus,

$$\mathscr{P}^{\leqslant 0} = \bigcup_{\gamma \in \mathbb{Q}_{>0}} \overline{\mathbb{Q}}[z^{-\gamma}].$$

The second step of the proof of Theorem 27 is the following result.

Proposition 34. *Consider a block upper triangular matrix $A \in \mathrm{GL}_d(\mathscr{P})$ of the form*

$$A = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_s \end{pmatrix} \tag{30}$$

for some $(A_1, \ldots, A_s) \in \mathrm{GL}_{r_1}(\overline{\mathbb{Q}}) \times \cdots \times \mathrm{GL}_{r_s}(\overline{\mathbb{Q}})$. There exists $F \in \mathfrak{H}_{\boldsymbol{r}}$ such that the coefficients of

$$F[A] = \begin{pmatrix} A_1 & & \star \\ & \ddots & \\ 0 & & A_s \end{pmatrix} \tag{31}$$

belong to $\mathscr{P}^{\leqslant 0}$.

The proof is given below, after the following preliminary result.

Lemma 35. *Consider $A_1, A_2 \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ and $B \in \mathrm{M}_d(\mathscr{P}^{>0})$. There exists $F \in \mathrm{M}_d(\mathscr{P}^{>0})$ such that*

$$F(z^p)A_1 - A_2F(z) = B(z). \tag{32}$$

Proof. We consider

$$F = -\sum_{k \geqslant 0} A_2^{-k-1} B(z^{p^k}) A_1^k \in \mathrm{M}_d(\mathscr{P}^{>0}).$$

A straightforward calculation shows that F satisfies (32). □

In the following proof, for any $F = \sum_{\gamma \in \mathbb{Q}} F_\gamma z^\gamma \in \mathrm{M}_d(\mathscr{H})$, we set

$$F^0 = F_0, \qquad F^{<0} = \sum_{\gamma \in \mathbb{Q}_{<0}} F_\gamma z^\gamma, \qquad F^{>0} = \sum_{\gamma \in \mathbb{Q}_{>0}} F_\gamma z^\gamma, \tag{33}$$

so that

$$F = F^{<0} + F^0 + F^{>0}.$$

Proof of Proposition 34. For any $(i,j) \in \{1, \dots, s\}^2$ with $j > i$ and any $M \in \mathrm{M}_{r_i, r_j}(\mathscr{P})$, we consider the matrix

$$T_{i,j}(M) = \begin{pmatrix} I_{r_1} & & 0 & & & & 0 \\ & \ddots & & & & & \\ & & I_{r_i} & & M & & \\ & 0 & & \ddots & & & 0 \\ & & & & I_{r_j} & & \\ & & & & & \ddots & \\ & & & & & & I_{r_s} \end{pmatrix} \in \mathfrak{H}_{\boldsymbol{r}}.$$

We have

$$T_{i,j}(M)^{-1} = T_{i,j}(-M).$$

We let $\mathscr{E}$ be the set of block upper triangular matrices $B \in \mathrm{GL}_d(\mathscr{P})$ whose diagonal blocks are $A_1, \dots, A_s$, that is, matrices of the form

$$B = \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_i & & B_{i,j} & & \\ & 0 & & \ddots & & & \\ & & & & A_j & & \\ & & & & & \ddots & \\ & & & & & & A_s \end{pmatrix} \in \mathrm{GL}_d(\mathscr{P}). \tag{34}$$

Note that $A \in \mathcal{E}$. For any $B \in \mathcal{E}$, the matrix $T_{i,j}(M)[B]$ belongs to $\mathcal{E}$ and its (k,l)th block is equal to

$$\begin{cases} -A_i M(z) + B_{i,j} + M(z^p)A_j & \text{if } (k,l)=(i,j), \\ B_{i,l} + M(z^p)B_{j,l} & \text{if } k=i \text{ and } l>j, \\ -B_{k,i}M(z) + B_{k,j} & \text{if } k<i \text{ and } l=j, \\ B_{k,l} & \text{else,} \end{cases} \tag{35}$$

i.e., zooming at the upper right corner of the matrix $T_{i,j}(M)[B]$ we obtain

$$\left. \begin{matrix} \cdots & * & \overbrace{-B_{1,i}M(z)+B_{1,j}}^{j\text{th column}} & * & & * \\ & & \vdots & & & \\ \cdots & * & -B_{i-1,i}M(z)+B_{i-1,j} & * & & * \\ & i\text{th row} \to & -A_iM(z)+B_{i,j}+M(z^p)A_j & B_{i,j+1}+M(z^p)B_{j,j+1} & \cdots & B_{i,s}+M(z^p)B_{j,s} \\ \cdots & * & * & * & & * \\ & \vdots & \vdots & \vdots & & \vdots \end{matrix} \right|$$

where the $*$-blocks are equal to the corresponding blocks of B.

We equip the set $\mathcal{S} := \{(k,l) \in \{1,\dots,s\}^2 \mid k < l\}$ with the following total order: $(k,l) \prec (k',l')$ if either ($l = l'$ and $k' < k$) or ($l < l'$). With respect to this order we have

$$(1,2) \prec (2,3) \prec (1,3) \prec (3,4) \prec (2,4) \prec (1,4) \prec (4,5) \prec (3,5) \prec \cdots$$

Then, we shall construct, for all $(k,l) \in \mathcal{S}$, a matrix $B^{[k,l]} \in \mathcal{E}$ satisfying the following property:

1. the systems $\phi_p Y = AY$ and $\phi_p Y = B^{[k,l]}Y$ are $\mathscr{P}$-equivalent *via* a gauge transformation in $\mathfrak{H}_{\boldsymbol{r}}$;
2. for all $(i,j) \in \mathcal{S}$ with $(i,j) \prec (k,l)$ or $(i,j) = (k,l)$, the (i,j)-block $(B^{[k,l]})_{i,j}$ of $B^{[k,l]}$ has coefficients in $\mathscr{P}^{\leqslant 0}$.

Our construction is recursive with respect to $\prec$ and proceeds as follows.

Construction of $B^{[1,2]}$. According to Lemma 35, there exists $M^{[1,2]} \in \mathrm{M}_{r_1,r_2}(\mathscr{P}^{>0})$ such that $M^{[1,2]}(z^p)A_2 - A_1 M^{[1,2]} = -A_{1,2}^{>0}$. Then, using (35), we see that the matrix

$$B^{[1,2]} = T_{1,2}(M^{[1,2]})[A]$$

belongs to $\mathcal{E}$ and that

$$(B^{[1,2]})_{1,2} = -A_1 M^{[1,2]} + A_{1,2} + M^{[1,2]}(z^p)A_2 = A_{1,2}^{\leqslant 0}$$

has coefficients in $\mathscr{P}^{\leqslant 0}$.

Construction of $B^{[2,3]}$ from $B^{[1,2]}$. According to Lemma 35, there exists $M^{[2,3]} \in \mathrm{M}_{r_2,r_3}(\mathscr{P}^{>0})$ such that $M^{[2,3]}(z^p)A_3 - A_2 M^{[2,3]} = -(B^{[1,2]})_{2,3}^{>0}$. Then, using (35), we see that the matrix

$$B^{[2,3]} = T_{2,3}(M^{[2,3]})[B^{[1,2]}]$$

belongs to $\mathscr{E}$ and that the matrices

$$\begin{array}{rcl}(B^{[2,3]})_{1,2} & = & (B^{[1,2]})_{1,2},\\ (B^{[2,3]})_{2,3} & = & -A_2M^{[2,3]}(z)+(B^{[1,2]})_{2,3}+M^{[2,3]}(z^p)A_3=(B^{[1,2]})_{2,3}^{\leqslant 0}\end{array}$$

have coefficients in $\mathscr{P}^{\leqslant 0}$.

Construction of $B^{[1,3]}$ from $B^{[2,3]}$. According to Lemma 35, there exists $M^{[1,3]}\in \mathrm{M}_{r_1,r_3}(\mathscr{P}^{>0})$ such that $M^{[1,3]}(z^p)A_3-A_1M^{[1,3]}=-(B^{[2,3]})_{1,3}^{>0}$. Then, using (35), we see that the matrix

$$B^{[1,3]}=T_{1,3}(M^{[1,3]})[B^{[2,3]}]$$

belongs to $\mathscr{E}$ and that the matrices

$$\begin{array}{rcl}(B^{[1,3]})_{1,2} & = & (B^{[2,3]})_{1,2},\\ (B^{[1,3]})_{2,3} & = & (B^{[2,3]})_{2,3},\\ (B^{[1,3]})_{1,3} & = & -A_1M^{[1,3]}+(B^{[2,3]})_{1,3}+M^{[1,3]}(z^p)A_3=(B^{[2,3]})_{1,3}^{\leqslant 0}\end{array}$$

have coefficients in $\mathscr{P}^{\leqslant 0}$.

We construct the other matrices $B^{[3,4]}$, $B^{[2,4]}$, $B^{[1,4]}$, $B^{[4,5]}$, $B^{[3,5]}$, ..., $B^{[s-1,s]}$ (in this order) is a similar way. It is clear that these matrices satisfy conditions (1) and (2) above.

In particular, the systems $\phi_pY=AY$ and $\phi_pY=B^{[s-1,s]}Y$ are $\mathscr{P}$-equivalent *via* a gauge transformation in $\mathfrak{H}_{\boldsymbol{r}}$ and $B^{[s-1,s]}$ is of the form (31). This concludes the proof.

□

4.5. **Third step of the proof of Theorem 27: elimination of the part of negative valuation.** Consider $s\in\mathbb{Z}_{\geqslant 1}$ and $\boldsymbol{r}=(r_1,\dots,r_s)\in\mathbb{Z}_{\geqslant 1}^s$ such that $r_1+\cdots+r_s=d$. We recall that $\mathfrak{V}_{\boldsymbol{r}}$ denotes the group of matrices of the form

$$F=\begin{pmatrix} I_{r_1} & & & & & & \\ & \ddots & & & & & \\ & & I_{r_i} & & F_{i,j} & & \\ & 0 & & \ddots & & & \\ & & & & I_{r_j} & & \\ & & & & & \ddots & \\ & & & & & & I_{r_s}\end{pmatrix}\in \mathrm{GL}_d(\mathscr{H}) \tag{36}$$

where $F_{i,j}\in \mathrm{M}_{r_i,r_j}(\mathscr{V}_{j-i})$. We continue with the notation (33).

The third step of the proof of Theorem 27 is the following result.

Proposition 36. *Consider a block upper triangular matrix*

$$A = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_s \end{pmatrix} \in \mathrm{GL}_d(\mathscr{P}) \cap \mathrm{M}_d(\mathscr{P}^{\leqslant 0}) \tag{37}$$

where $(A_1, \ldots, A_s) \in \mathrm{GL}_{r_1}(\overline{\mathbb{Q}}) \times \cdots \times \mathrm{GL}_{r_s}(\overline{\mathbb{Q}})$. *Then, there exists* $F \in \mathfrak{V}_{\boldsymbol{r}}$ *such that* $F[A] = A^0$.

We recall that A^0 denotes the constant coefficient of A seen as a Puiseux series with coefficients in $\mathrm{M}_d(\overline{\mathbb{Q}})$. The proof of Proposition 36 is given below, after a preliminary result.

Lemma 37. *Consider* $A_1, A_2 \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ *and* $B \in \mathrm{M}_d(\mathcal{V}_s)$. *There exists* $F(z) \in \mathrm{M}_d(\mathcal{V}_{s+1})$ *such that*

$$F(z^p)A_1 - A_2F(z) = B(z). \tag{38}$$

Proof. By linearity, it is sufficient to treat the case

$$B(z) = h(z)R$$

with $R \in \mathrm{M}_d(\overline{\mathbb{Q}})$ and $h \in \mathcal{V}_s$. Since $h \in \mathcal{V}_s \subset \mathscr{H}^{<0}$, it follows from Lemma 16 that the family $(h(z^{\frac{1}{p^k}})A_2^{k-1}RA_1^{-k})_{k\geqslant 1}$ is summable. So, we can consider

$$F(z) = \sum_{k\geqslant 1} h(z^{\frac{1}{p^k}})A_2^{k-1}RA_1^{-k} \in \mathrm{M}_d(\mathscr{H}).$$

We claim that this F has the desired properties. Indeed, the following calculation shows that F satisfies (38):

$$\begin{aligned} & F(z^p)A_1 - A_2F(z) \\ = \; & \sum_{k\geqslant 1} h(z^{\frac{1}{p^{k-1}}})A_2^{k-1}RA_1^{-k}A_1 - A_2\sum_{k\geqslant 1} h(z^{\frac{1}{p^k}})A_2^{k-1}RA_1^{-k} \\ = \; & \sum_{k\geqslant 1} h(z^{\frac{1}{p^{k-1}}})A_2^{k-1}RA_1^{-(k-1)} - \sum_{k\geqslant 1} h(z^{\frac{1}{p^k}})A_2^{k}RA_1^{-k} \\ = \; & h(z^{\frac{1}{p^{1-1}}})A_2^{1-1}RA_1^{-(1-1)} \\ = \; & B(z). \end{aligned}$$

It remains to prove that the coefficients of F belong to $\mathcal{V}_{s+1}$. We let $A_1^{-1} = D_1 + N_1$ and $A_2 = D_2 + N_2$ be the Dunford-Jordan decomposition of A_1^{-1} and A_2 respectively (*i.e.*, for all $i \in \{1, 2\}$, D_i is semisimple, N_i is nilpotent and $D_iN_i = N_iD_i$). Using the Newton binomial formula

$$A_1^{-k} = \sum_{l=0}^{k} \binom{k}{l} D_1^{k-l}N_1^l = \sum_{l=0}^{d-1} \binom{k}{l} N_1^l D_1^{k-l}$$

and

$$A_2^{k} = \sum_{l=0}^{k} \binom{k}{l} D_2^{k-l}N_2^l = \sum_{l=0}^{d-1} \binom{k}{l} D_2^{k-l}N_2^l$$

(where $\binom{k}{l} = 0$ if $l > k$), we see that F is a $\overline{\mathbb{Q}}$-linear combinaison of terms of the form

$$\sum_{k\geqslant 1} k^{\alpha} h(z^{\frac{1}{p^k}}) D_2^{k-1} S D_1^k \tag{39}$$

with $S \in \mathrm{M}_d(\overline{\mathbb{Q}})$ and $\alpha \in \mathbb{Z}_{\geqslant 0}$. In order to conclude the proof, it remains to prove that the entries of (39) are in $\mathcal{V}_{s+1}$. Since the latter property is invariant by right and left multiplication by an element of $\mathrm{GL}_d(\overline{\mathbb{Q}})$, we can assume that the D_i are diagonal, say $D_i = \mathrm{diag}(c_{i,1}, \ldots, c_{i,d})$. In that case, setting $S = (s_{i,j})_{1\leqslant i,j\leqslant n}$, we have

$$(39) = \left(s_{i,j} \sum_{k\geqslant 1} k^{\alpha} c_{2,i}^{k-1} c_{1,j}^{k} h(z^{\frac{1}{p^k}}) \right)_{1\leqslant i,j\leqslant d}.$$

It follows from Lemma 19 that the entries of the latter matrix are in $\mathcal{V}_{s+1}$ as expected. □

Proof of Proposition 36. We set

$$A = \begin{pmatrix} A_1 & & & & & & \\ & \ddots & & & & & \\ & & A_i & & A_{i,j} & & \\ & 0 & & \ddots & & & \\ & & & & A_j & & \\ & & & & & \ddots & \\ & & & & & & A_s \end{pmatrix} \in \mathrm{GL}_d(\mathscr{P})$$

where $A_{i,j} \in \mathrm{M}_{r_i,r_j}(\mathscr{P}^{\leqslant 0})$.

We want to prove that there exists $F \in \mathfrak{V}_{\boldsymbol{r}}$ such that $F[A] = A^0$, *i.e.*, such that $F(z^p)A(z) = A^0 F(z)$. Note that, for any

$$F = \begin{pmatrix} I_{r_1} & & & & & & \\ & \ddots & & & & & \\ & & I_{r_i} & & F_{i,j} & & \\ & 0 & & \ddots & & & \\ & & & & I_{r_j} & & \\ & & & & & \ddots & \\ & & & & & & I_{r_s} \end{pmatrix} \in \mathfrak{V}_{\boldsymbol{r}}, \tag{40}$$

the equation $F[A] = A^0$ is equivalent to: for all $(i,j) \in \{1,\ldots,s\}^2$ with $j \geqslant i$,

$$\sum_{k=i}^{j} F_{i,k}(z^p) A_{k,j}(z) = \sum_{k=i}^{j} A^0_{i,k} F_{k,j}(z).$$

This can be rewritten as follows: for all $(i,j) \in \{1,\ldots,s\}^2$ with $j \geqslant i$,

$$(41)\quad F_{i,j}(z^p)A_j - A_iF_{i,j}(z) = \sum_{k=i+1}^{j} A^0_{i,k}F_{k,j}(z) - \sum_{k=i}^{j-1} F_{i,k}(z^p)A_{k,j}(z)$$
$$= (A^0_{i,j} - A_{i,j}(z)) + \sum_{k=i+1}^{j-1} A^0_{i,k}F_{k,j}(z) - \sum_{k=i+1}^{j-1} F_{i,k}(z^p)A_{k,j}(z)$$

(where the sums $\sum_{k=i+1}^{j-1}$ are 0 if $j-1 < i+1$). We shall now prove the existence of such $F_{i,j} \in \mathrm{M}_{r_i,r_j}(\mathcal{V}_{j-i})$ by induction on $\delta = j-i \in \{0,\ldots,s-1\}$.

Base case $\delta = 0$. The case $\delta = 0$ corresponds to $i = j$ and, in this case, we can set $F_{i,j}(z) = F_{i,i}(z) = I_{r_i}$.

Inductive step $\delta \to \delta+1$. Consider $\delta \in \{0,\ldots,s-2\}$ and assume that we can find $F_{i,j}(z) \in \mathrm{M}_{r_i,r_j}(\mathcal{V}_{j-i})$ for all $i,j \in \{1,\ldots,s\}$ with $j \geqslant i$ such that $j-i \leqslant \delta$. Then, for any $i,j \in \{1,\ldots,s\}$ with $j \geqslant i$ such that $j-i = \delta+1$, the terms involved in the right hand side of (41) are known and this right hand side has coeffcients in $\mathcal{V}_\delta$. Now, Lemma 37 ensures that, for any $i,j \in \{1,\ldots,s\}$ with $j \geqslant i$ such that $j-i = \delta+1$, we can find $F_{i,j}(z) \in \mathrm{M}_{r_i,r_j}(\mathcal{V}_{\delta+1})$ satisfying (41). This concludes the induction. ☐

4.6. **Conclusion of the proof of Theorem 27.** Proposition 30 guaranties that there exist $s \in \mathbb{Z}_{\geqslant 1}$, $\boldsymbol{r} = (r_1,\ldots,r_s) \in \mathbb{Z}^s_{\geqslant 1}$ such that $r_1+\cdots+r_s = d$, $(A_1,\ldots,A_s) \in \mathrm{GL}_{r_1}(\overline{\mathbb{Q}}) \times \cdots \times \mathrm{GL}_{r_s}(\overline{\mathbb{Q}})$ and $G \in \mathrm{GL}_d(\mathscr{P})$ such that

$$(42)\qquad G[A] = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_s \end{pmatrix}.$$

Proposition 34 ensures that there exists $H \in \mathfrak{H}_{\boldsymbol{r}}$ such that the coefficients of

$$H[G[A]] = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_i \end{pmatrix}$$

belong to $\mathscr{P}^{\leqslant 0} = \bigcup_{\gamma \in \mathbb{Q}_{>0}} \overline{\mathbb{Q}}[z^{-\gamma}]$. Proposition 36 guaranties that there exists $K \in \mathfrak{V}_{\boldsymbol{r}}$ such that

$$K[H[G[A]]] = C$$

where $C \in \mathrm{GL}_d(\overline{\mathbb{Q}})$ is the coefficient of z^0 in $H[G[A]]$.

Then, $F_1 = (HG)^{-1}$, $F_2 = K^{-1}$, $\Theta = H[G[A]]$ and the matrix C defined above have the properties required by Theorem 27.

4.7. **Proof of Theorem 29.** We let F_1, F_2, C, Θ be the matrices given by Theorem 27 and we set $F = F_1F_2$, so that Fe_C is a fundamental matrix of solutions of (21) and that

$$F_2(z^p)C = \Theta(z)F_2(z).$$

Consider another fundamental matrix of solutions of (21) of the form $F'e_{C'}$ with $F' \in \mathrm{GL}_d(\mathscr{H})$ and $C' \in \mathrm{GL}_d(\overline{\mathbb{Q}})$. Combining the facts that Fe_C (resp. $F'e_{C'}$) is solution of (21) and the equality $\phi_p(e_C) = Ce_C$, we get $F(z^p)C = A(z)F(z)$ (resp. $F'(z^p)C' = A(z)F'(z)$). Setting $R = F^{-1}F' \in \mathrm{GL}_d(\mathscr{H})$, we get $R(z^p)C' = CR(z)$. Setting $R = \sum_{\gamma\in\mathbb{Q}} r_\gamma z^\gamma$, we obtain, for all $\gamma \in \mathbb{Q}$, $r_{\frac{\gamma}{p}}C' = Cr_\gamma$. This implies that the support $\mathrm{supp}(R)$ of R is left invariant by multiplication by $p^{\mathbb{Z}}$. Since $\mathrm{supp}(R)$ is well-ordered, this implies that $\mathrm{supp}(R) \subset \{0\}$, *i.e.*, $R \in \mathrm{GL}_d(\overline{\mathbb{Q}})$. Thus, $F' = FR$ and $C' = R^{-1}CR$. Now set $F_2' = F_1^{-1}F'$. Then, $F' = F_1F_2'$, $F_2' = F_2R$ and

$$F_2'(z^p)C' = F_2(z^p)RC' = F_2(z^p)CR = \Theta(z)F_2(z)R = \Theta(z)F_2'(z)\,.$$

Thus, the matrix R has the desired properties.

4.8. **Proof of Theorem 4.** Theorem 4 states that the p-Mahler equation (1) has a full basis of generalized p-Mahler series solutions, *i.e.*, it has d $\overline{\mathbb{Q}}$-linearly independent generalized p-Mahler series solutions $y_1, \ldots, y_d \in \mathscr{R}$. To prove this, we consider the p-Mahler system associated to equation (1), namely

$$Y(z^p) = A(z)Y(z) \tag{43}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{a_0}{a_d} & -\frac{a_1}{a_d} & & & -\frac{a_{d-1}}{a_d} \end{pmatrix}. \tag{44}$$

It follows from Theorem 27 that this system has a fundamental matrix of solutions of the form

$$F = F_1F_2e_C$$

where

- $F_1 \in \mathrm{GL}_d(\mathscr{P})$ satisfies $\phi_p(F_1)\Theta = AF_1$, for some matrix $\Theta \in \mathrm{GL}_d(\mathbb{K}_\infty)$;
- the entries of $F_2 \in \mathrm{GL}_d(\mathscr{H})$ belong to $\mathcal{V}$;
- the entries of e_C are $\overline{\mathbb{Q}}$-linear combinaisons of elements of the form $e_c\ell^j$ with $c \in \mathrm{Sp}(C)$ and $j \in \{0, \ldots, d-1\}$ (see Lemma 24).

Note that the identity $\phi_p(F_1) = AF_1\Theta^{-1}$ implies that the finite dimensional $\mathbb{K}_\infty$-vector space spanned by the entries of F_1 is invariant under ϕ_p; this implies that the entries of F_1 are p-Mahler Puiseux series.

Now, the entries $y_1, \ldots, y_d$ of the first row of F are $\overline{\mathbb{Q}}$-linearly independent solutions of (1) and it follows from the properties of the entries of F_1, F_2 and e_C listed above that they have the form

$$y_i = \sum_{c\in\mathrm{Sp}(C), j\in\{0,\ldots,d-1\}} f_{c,j}e_c\ell^j$$

where the $f_{c,j}$ are finite sums of products of a p-Mahler Puiseux series by an element of $\mathcal{V}$. Since any element of $\mathcal{V}$ is a linear combination over $\overline{\mathbb{Q}}[z^{-\frac{1}{*}}]$ of some $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$, the $f_{c,j}$ are finite sums of products of a p-Mahler Puiseux series with a Hahn series of the form $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$. Hence, the y_i are generalized p-Mahler series. This concludes the proof.

5. Standard decomposition: proofs of Propositions 6 and 10

Recall that, according to Definition 5, the decomposition (3)-(4) of a generalized p-Mahler series f is standard if the triples $(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a})$ involved in the support of the sum in (4) belong to the set

$$\boldsymbol{\Lambda}_{\mathrm{st}} = \bigcup_{t \in \mathbb{Z}_{\geqslant 0}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^\times)^t \times \mathbb{N}_{(p)}^t$$

where $\mathbb{N}_{(p)}$ is the set of positive rational numbers whose denominator is coprime with p and whose numerator is not divisible by p.

5.1. Proof of Proposition 6.

Proposition 6 states that any generalized p-Mahler series has a unique standard decomposition. The uniqueness is proved in Section 5.1.1, the existence is proved in Section 5.1.2.

5.1.1. *Uniqueness.* The uniqueness of the standard decomposition in Proposition 6 is clearly implied by the following result.

Proposition 38. *The family*

$$(\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j)_{(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \boldsymbol{\Lambda}_{\mathrm{st}}, (c,j) \in \overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}}$$

is $\mathscr{P}$-linearly independent.

Proof. We first note that [Roq24, Lemma 30] guaranties that the family $(e_c \ell^j)_{(c,j) \in \overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}}$ is $\mathscr{H}$-linearly independent. Therefore, in order to conclude the proof, it is sufficient to prove that the family

$$(\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}})_{(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \boldsymbol{\Lambda}_{\mathrm{st}}}$$

is $\mathscr{P}$-linearly independent. This is ensured by Proposition 39 below. □

Proposition 39. *The family*

$$(\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}})_{(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \boldsymbol{\Lambda}_{\mathrm{st}}}$$

is $\mathscr{P}$-linearly independent.

This result is proved below after the following lemma.

Lemma 40. *Consider $s, t \in \mathbb{Z}_{\geqslant 0}$ such that $s \geqslant t \geqslant 0$, $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{N}_{(p)}^s$, $\mathbf{b} = (b_1, \ldots, b_t) \in \mathbb{N}_{(p)}^t$, $d \in \mathbb{Z}_{\geqslant 1}$. Then, there exists $C_{\mathbf{a},\mathbf{b},d} > 0$ such that, for all $\gamma \in \frac{1}{d}\mathbb{Z}$, for all $k_1, \ldots, k_s, \ell_1, \ldots, \ell_t \in \mathbb{Z}_{\geqslant 1}$ such that $k_1, \ldots, k_s \geqslant C_{\mathbf{a},\mathbf{b},d}$, we have*

$$\frac{a_1}{p^{k_1}} + \frac{a_2}{p^{k_1+k_2}} + \cdots + \frac{a_s}{p^{k_1+k_2+\cdots+k_s}} = \gamma + \frac{b_1}{p^{\ell_1}} + \frac{b_2}{p^{\ell_1+\ell_2}} + \cdots + \frac{b_t}{p^{\ell_1+\ell_2+\cdots+\ell_t}} \tag{45}$$

if and only if $\gamma = 0$, $s = t$ and, for all $i \in \{1, \ldots, t\}$, $a_i = b_i$ and $k_i = \ell_i$.

Proof. We first consider the case $t = 0$.

Case $t = 0$**.** We have to prove that, for any $s \in \mathbb{Z}_{\geqslant 0}$, $\mathbf{a} = (a_1, \ldots, a_s) \in \mathbb{N}_{(p)}^s$ and $d \in \mathbb{Z}_{\geqslant 1}$, there exists $C_{\mathbf{a},d} > 0$ such that, for all $\gamma \in \frac{1}{d}\mathbb{Z}$, for all $k_1, \ldots, k_s \in \mathbb{Z}_{\geqslant 0}$ such that $k_1, \ldots, k_s \geqslant C_{\mathbf{a},d}$, the equality

$$\frac{a_1}{p^{k_1}} + \frac{a_2}{p^{k_1+k_2}} + \cdots + \frac{a_s}{p^{k_1+k_2+\cdots+k_s}} = \gamma \tag{46}$$

implies $s = 0$ and $\gamma = 0$. We claim that any $C_{\mathbf{a},d} \in \mathbb{Z}_{\geqslant 1}$ such that

$$\frac{a_1 + \cdots + a_s}{p^{C_{\mathbf{a},d}}} < \frac{1}{d}$$

works. Indeed, the latter inequality implies that the left-hand side of (46) belongs to $[0, \frac{1}{d}[$. Since the right-hand side of (46) belongs to $\frac{1}{d}\mathbb{Z}$, we get that both sides or (46) are equal to 0, so $s = 0$ and $\gamma = 0$.

Let us now turn to the proof of the result in the general case. We argue by induction on $n = s + t$.

The result is already proved when $n = 0$ or $n = 1$ for, in that case, we have $t = 0$.

Consider an integer $n \geqslant 2$. We assume that the result is true for all $s \geqslant t \geqslant 0$ such that $s + t \leqslant n - 1$. Consider $s \geqslant t \geqslant 0$ such that $s + t = n$ and let us prove that the result is true for these s, t.

We already know that the result is true if $t = 0$, so we can and will assume that $t \geqslant 1$. We distinguish two cases: $a_s = b_t$ and $a_s \neq b_t$.

Case $a_s = b_t$**.** Let $D \in \mathbb{Z}_{\geqslant 1}$ be such that p is coprime with the denominator of $p^D \frac{1}{d}$. Consider $\mathbf{a'} = (a_1, \ldots, a_{s-1})$, $\mathbf{b'} = (b_1, \ldots, b_{t-1})$ and the corresponding constant $C_{\mathbf{a'},\mathbf{b'},d}$ given by the induction hypothesis. We set $C_{\mathbf{a},\mathbf{b},d} = \max\{D+1, C_{\mathbf{a'},\mathbf{b'},d}\}$.

Consider $\gamma \in \frac{1}{d}\mathbb{Z}$ and $k_1, \ldots, k_s, \ell_1, \ldots, \ell_t \in \mathbb{Z}_{\geqslant 1}$ such that $k_1, \ldots, k_s \geqslant C_{\mathbf{a},\mathbf{b},d}$ satisfying (45). We rewrite the equality (45) as follows

$$\frac{a_1}{p^{k_1}} + \frac{a_2}{p^{k_1+k_2}} + \cdots + \frac{a_s}{p^{k_1+k_2+\cdots+k_s}} - \gamma = \frac{b_1}{p^{\ell_1}} + \frac{b_2}{p^{\ell_1+\ell_2}} + \cdots + \frac{b_t}{p^{\ell_1+\ell_2+\cdots+\ell_t}} \tag{47}$$

The left-hand side of (47) belongs to $\frac{1}{p^{k_1+k_2+\cdots+k_s}}\mathbb{N}_{(p)}$ whereas its right-hand side belongs to $\frac{1}{p^{\ell_1+\ell_2+\cdots+\ell_t}}\mathbb{N}_{(p)}$. So, we have

$$k_1 + k_2 + \cdots + k_s = \ell_1 + \ell_2 + \cdots + \ell_t. \tag{48}$$

Thus, $\frac{a_s}{p^{k_1+k_2+\cdots+k_s}} = \frac{b_t}{p^{\ell_1+\ell_2+\cdots+\ell_t}}$ and (45) gives

$$\frac{a_1}{p^{k_1}} + \frac{a_2}{p^{k_1+k_2}} + \cdots + \frac{a_{s-1}}{p^{k_1+k_2+\cdots+k_{s-1}}} = \gamma + \frac{b_1}{p^{\ell_1}} + \frac{b_2}{p^{\ell_1+\ell_2}} + \cdots + \frac{b_{t-1}}{p^{\ell_1+\ell_2+\cdots+\ell_{t-1}}}.$$

Since $k_1, \ldots, k_{s-1} \geqslant C_{\mathbf{a'},\mathbf{b'},d}$, we have $\gamma = 0$, $s - 1 = t - 1$, $a_i = b_i$ and $k_i = \ell_i$ for all $i \in \{1, \ldots, t-1\}$. It follows from (48) that $k_s = \ell_s$ as well. Moreover, we have $a_s = b_s$ by hypothesis. This concludes the induction step in this case.

Case $a_s \neq b_t$. We first treat the case $t=1$. Let $L \in \mathbb{Z}_{\geqslant 0}$ be such that p^L is the greatest power of p dividing the numerator of $a_s - b_1$. Let $D \in \mathbb{Z}_{\geqslant 1}$ be such that the numerators of $p^D \frac{1}{d}$, $p^D a_1, \ldots, p^D a_{s-1}$ is divisible by p^L. We set $C_{\mathbf{a},\mathbf{b},d} = D+1$.

We claim that there is no $\gamma \in \frac{1}{d}\mathbb{Z}$ and $k_1, \ldots, k_s, \ell_1 \in \mathbb{Z}_{\geqslant 0}$ such that $k_1, \ldots, k_s \geqslant C_{\mathbf{a},\mathbf{b},d}$ satisfying (45). We argue by contradiction: we assume that such $k_1, \ldots, k_s, \ell_1$ exist. Arguing as in the case $a_s = b_t$ treated above, we see that

$$k_1 + k_2 + \cdots + k_s = \ell_1.$$

Multiplying both sides of (45) by $p^{k_1+\cdots+k_s} = p^{\ell_1}$, we obtain

$$p^{k_2+\cdots+k_s} a_1 + \cdots + p^{k_s} a_{s-1} + a_s = p^{k_1+\cdots+k_s}\gamma + b_1. \tag{49}$$

This can be rewritten as follows

$$0 = (a_s - b_1) + p^{k_2+\cdots+k_s} a_1 + \cdots + p^{k_s} a_{s-1} - p^{k_1+\cdots+k_s}\gamma. \tag{50}$$

But, by our choice of $C_{\mathbf{a},\mathbf{b},d}$, p^{L+1} divides the numerator of $p^{k_2+\cdots+k_s} a_1 + \cdots + p^{k_s} a_{s-1} - p^{k_1+\cdots+k_s}\gamma$ but not the numerator of $a_s - b_1$. This is in contradiction with (50).

We now treat the case $t \geqslant 2$. Let $L \in \mathbb{Z}_{\geqslant 0}$ be such that p^L is the greatest power of p dividing the numerator of $a_s - b_t$. Let $D \in \mathbb{Z}_{\geqslant 1}$ be such that the numerators of $p^D \frac{1}{d}$, $p^D a_1, \ldots, p^D a_{s-1}$ is divisible by p^L. For any $\ell \in \{0, \ldots, L\}$, consider $\mathbf{a} = (a_1, \ldots, a_s)$, $\mathbf{b}_\ell = (b_1, \ldots, b_{t-2}, p^\ell b_{t-1} + b_t)$ and the corresponding constant $C_{\mathbf{a},\mathbf{b}_\ell,d}$ given by the induction hypothesis. We set $C_{\mathbf{a},\mathbf{b},d} = \max\{D+1, C_{\mathbf{a},\mathbf{b}_0,d}, \ldots, C_{\mathbf{a},\mathbf{b}_L,d}\}$.

We claim that there is no $\gamma \in \frac{1}{d}\mathbb{Z}$ and $k_1, \ldots, k_s, \ell_1, \ldots, \ell_t \in \mathbb{Z}_{\geqslant 0}$ such that $k_1, \ldots, k_s \geqslant C_{\mathbf{a},\mathbf{b},d}$ satisfying (45). We argue by contradiction: we assume that such $k_1, \ldots, k_s, \ell_1, \ldots, \ell_t$ exist. Arguing as in the case $a_s = b_t$ treated above, we see that

$$k_1 + k_2 + \cdots + k_s = \ell_1 + \ell_2 + \cdots + \ell_t.$$

Multiplying both sides of (45) by $p^{k_1+\cdots+k_s} = p^{\ell_1+\cdots+\ell_t}$, we obtain

$$p^{k_2+\cdots+k_s} a_1 + \cdots + p^{k_s} a_{s-1} + a_s = p^{k_1+\cdots+k_s}\gamma + p^{\ell_2+\cdots+\ell_t} b_1 + \cdots + p^{\ell_t} b_{t-1} + b_t. \tag{51}$$

This can be rewritten as follows

$$p^{\ell_2+\cdots+\ell_t} b_1 + \cdots + p^{\ell_t} b_{t-1} = (a_s - b_t) + p^{k_2+\cdots+k_s} a_1 + \cdots + p^{k_s} a_{s-1} - p^{k_1+\cdots+k_s}\gamma. \tag{52}$$

Since p^{L+1} divides the numerator of $p^{k_2+\cdots+k_s} a_1 + \cdots + p^{k_s} a_{s-1} - p^{k_1+\cdots+k_s}\gamma$ but not the numerator of $a_s - b_t$, we get that $\ell_t \in \{0, \ldots, L\}$. Rewriting (45) as follows

$$\frac{a_1}{p^{k_1}} + \cdots + \frac{a_s}{p^{k_1+k_2+\cdots+k_s}} = \gamma + \frac{b_1}{p^{\ell_1}} + \cdots + \frac{b_{t-2}}{p^{\ell_1+\ell_2+\cdots+\ell_{t-2}}} + \frac{p^{\ell_t} b_{t-1} + b_t}{p^{\ell_1+\ell_2+\cdots+\ell_t}}$$

and using the fact that $k_1, \ldots, k_s \geqslant C_{\mathbf{a},\mathbf{b}_{\ell_t},d}$, we get that $s = t-1$, which is absurd because $t \leqslant s$. □

Proof of Proposition 39. We want to prove that the family $(\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}})_{(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in\boldsymbol{\Lambda}_{\mathrm{st}}}$ is $\mathscr{P}$-linearly independent. Assume, on the contrary, that it is $\mathscr{P}$-linearly dependent. Then, there exist $r \in \mathbb{Z}_{\geqslant 1}$, pairwise distinct triples $(\boldsymbol{\alpha}_i, \boldsymbol{\lambda}_i, \boldsymbol{a}_i) \in \boldsymbol{\Lambda}_{\mathrm{st}}$, $1 \leqslant i \leqslant r$, and Puiseux series $f_1, \ldots, f_r \in \mathscr{P}^\times$ such that

$$\sum_{i=1}^{r} f_i \xi_{\boldsymbol{\alpha}_i,\boldsymbol{\lambda}_i,\boldsymbol{a}_i} = 0. \tag{53}$$

For any $i \in \{1, \ldots, r\}$, we write $\boldsymbol{a}_i = (a_{i,1}, \ldots, a_{i,t_i})$, $\boldsymbol{\alpha}_i = (\alpha_{i,1}, \ldots, \alpha_{i,t_i})$, $\boldsymbol{\lambda}_i = (\lambda_{i,1}, \ldots, \lambda_{i,t_i})$. Up to renumbering, we might suppose that $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_r$.

If $t_r = 0$, then $r = 1$ and $(\boldsymbol{\alpha}_1, \boldsymbol{\lambda}_1, \boldsymbol{a}_1) = ((), (), ())$. So, (53) reduces to $f_1 \xi_{(),(),()} = f_1 = 0$, whence a contradiction.

We now assume that $t_r \geqslant 1$. Up to multiplying (53) by some power of z, we might suppose that the z-adic valuation of f_r is 0. Let $d \in \mathbb{Z}_{\geqslant 1}$ be such that the supports of $f_1, \ldots, f_r$ are included in $\frac{1}{d}\mathbb{Z}$. We consider the constants $C_{\mathbf{a}_r,\mathbf{a}_1,d}, \ldots, C_{\mathbf{a}_r,\mathbf{a}_r,d}$ given by Lemma 40 and we set $C = \max\{C_{\mathbf{a}_r,\mathbf{a}_1,d}, \ldots, C_{\mathbf{a}_r,\mathbf{a}_r,d}\}$. So, for any $\gamma \in \frac{1}{d}\mathbb{Z}$, for any $i \in \{1, \ldots, r-1\}$, for any $k_1, \ldots, k_{t_r}, \ell_1, \ldots, \ell_{t_i} \in \mathbb{Z}_{\geqslant 1}$ such that $k_1, \ldots, k_{t_r} \geqslant C$, the equality

$$\frac{a_{r,1}}{p^{k_1}} + \cdots + \frac{a_{r,t_r}}{p^{k_1+\cdots+k_{t_r}}} = \gamma + \frac{a_{i,1}}{p^{\ell_1}} + \cdots + \frac{a_{i,t_i}}{p^{\ell_1+\cdots+\ell_{t_i}}}$$

holds if and only if $\gamma = 0$, $t_r = t_i$, $\mathbf{a}_r = \mathbf{a}_i$ and $(k_1, \ldots, k_{t_r}) = (\ell_1, \ldots, \ell_{t_r})$.

This implies that, for any $k_1, \ldots, k_{t_r} \in \mathbb{Z}_{\geqslant C}$, the coefficient of

$$z^{-\frac{a_{r,1}}{p^{k_1}} - \cdots - \frac{a_{r,t_r}}{p^{k_1+\cdots+k_{t_r}}}} \tag{54}$$

in $f_i \xi_{\boldsymbol{\alpha}_i,\boldsymbol{\lambda}_i,\boldsymbol{a}_i}$ is equal to:

- $c_i k_1^{\alpha_{i,1}} \cdots k_{t_r}^{\alpha_{i,t_r}} \lambda_{i,1}^{k_1} \cdots \lambda_{i,t_r}^{k_1+\cdots+k_{t_r}}$ if $\boldsymbol{a}_i = \boldsymbol{a}_r$ where $c_i \neq 0$ is the constant coefficient of f_i;
- 0 if $\boldsymbol{a}_i \neq \boldsymbol{a}_r$.

The equality (53) guaranties that the sum of the coefficients of (54) in $f_1 \xi_{\boldsymbol{\alpha}_1,\boldsymbol{\lambda}_1,\boldsymbol{a}_1}, \ldots, f_r \xi_{\boldsymbol{\alpha}_r,\boldsymbol{\lambda}_r,\boldsymbol{a}_r}$ is equal to 0, so, for any $k_1, \ldots, k_t \in \mathbb{Z}_{\geqslant C}$, we have

$$\sum_{i\in\mathcal{I}} c_i k_1^{\alpha_{i,1}} \cdots k_t^{\alpha_{i,t}} \lambda_{i,1}^{k_1} \cdots \lambda_{i,t}^{k_1+\cdots+k_t} = 0 \tag{55}$$

where $\mathcal{I}$ be the set of $i \in \{1, \ldots, r\}$ such that $\boldsymbol{a}_i = \boldsymbol{a}_r$ and where $t = t_r$ is the common value of the t_i for $i \in \mathcal{I}$. But, since the $2t$-uplets $(\boldsymbol{\alpha}_i, \boldsymbol{\lambda}_i)$ are pairwise distinct when i varies in $\mathcal{I}$, it follows from [Sch03, Lem. 2.2] that the family

$$\left((k_1^{\alpha_{i,1}} \cdots k_t^{\alpha_{i,t}} \lambda_{i,1}^{k_1} \cdots \lambda_{i,t}^{k_1+\cdots+k_t})_{(k_1,\ldots,k_t)\in\mathbb{Z}^t}\right)_{i\in\mathcal{I}}$$

is $\overline{\mathbb{Q}}$-linearly independent. Actually, a straightforward adaptation of the proof of [Sch03, Lem. 2.2] implies that the family

$$\left((k_1^{\alpha_{i,1}} \cdots k_t^{\alpha_{i,t}} \lambda_{i,1}^{k_1} \cdots \lambda_{i,t}^{k_1+\cdots+k_t})_{(k_1,\ldots,k_t)\in\mathbb{Z}^t_{\geqslant C}}\right)_{i\in\mathcal{I}}$$

is $\overline{\mathbb{Q}}$-linearly independent (see also [FR24b]). This contradicts (55).

□

5.1.2. *Existence.* In order to prove the existence of the standard decomposition stated in Proposition 6, it is clearly sufficient to prove that any $z^{-\gamma}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ is a $\overline{\mathbb{Q}}$-linear combinaison of terms of the form $z^{-\gamma'}\xi_{\boldsymbol{\alpha}',\boldsymbol{\lambda}',\boldsymbol{a}'}$ where $\boldsymbol{a}'$ has entries in $\mathbb{N}_{(p)}$. The latter property is true and follows immediately from the following lemma.

Lemma 41. *For any $s \in \mathbb{Z}_{\geqslant 0}$, the $\overline{\mathbb{Q}}$-vector space $\mathcal{V}_s$ defined in Section 2.2 admits the following alternative description:*

$$\mathcal{V}_s = \operatorname{Span}_{\overline{\mathbb{Q}}}\Big(\{z^{-\gamma}\xi_{(),(),()} \mid \gamma \in \mathbb{Q}_{>0}\} \cup \bigcup_{t\in\{1,\ldots,s\}} \{z^{-\gamma}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \mid \gamma \in \mathbb{Q}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \mathbb{Z}_{\geqslant 0}^{t} \times (\overline{\mathbb{Q}}^{\times})^{t} \times \mathbb{N}_{(p)}^{t}\}\Big). \tag{56}$$

Proof. Let $\mathcal{W}_s$ be the $\overline{\mathbb{Q}}$-vector space given by the right-hand side of (56). We want to prove that $\mathcal{V}_s = \mathcal{W}_s$. The inclusion $\mathcal{W}_s \subset \mathcal{V}_s$ is obvious. In order to prove the converse inclusion $\mathcal{V}_s \subset \mathcal{W}_s$, we argue by induction on s.

Base case $s = 0$. The inclusion $\mathcal{V}_s \subset \mathcal{W}_s$ for $s = 0$ is true because $\mathcal{V}_0 = \mathcal{W}_0 = \mathscr{P}^{<0}$.

Inductive step $s - 1 \to s$. Suppose that $\mathcal{V}_{s-1} \subset \mathcal{W}_{s-1}$ for some $s \in \mathbb{Z}_{\geqslant 1}$ and let us prove that $\mathcal{V}_s \subset \mathcal{W}_s$. We consider the nondecreasing filtration $(\mathcal{V}_{s,\beta})_{\beta\in\mathbb{Z}_{\geqslant -1}}$ of $\mathcal{V}_s$ given by the $\overline{\mathbb{Q}}$-vector spaces defined by

$$\mathcal{V}_{s,-1} = \mathcal{V}_{s-1}$$

and, for all $\beta \in \mathbb{Z}_{\geqslant 0}$,

$$\mathcal{V}_{s,\beta} = \mathcal{V}_{s-1} + \operatorname{Span}_{\overline{\mathbb{Q}}}\Big(\{z^{-\gamma}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \mid \gamma \in \mathbb{Q}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in (\{0,\ldots,\beta\} \times \mathbb{Z}_{\geqslant 0}^{s-1}) \times (\overline{\mathbb{Q}}^{\times})^{s} \times \mathbb{Q}_{>0}^{s}\}\Big).$$

Proving $\mathcal{V}_s \subset \mathcal{W}_s$ is equivalent to proving that, for all $\beta \in \mathbb{Z}_{\geqslant -1}$, $\mathcal{V}_{s,\beta} \subset \mathcal{W}_s$. Let us prove this by induction on β.

Base case $\beta = -1$. If $\beta = -1$, then $\mathcal{V}_{s,\beta} = \mathcal{V}_{s,-1} = \mathcal{V}_{s-1}$ and the inclusion $\mathcal{V}_{s,\beta} \subset \mathcal{W}_s$ follows from the inductive hypothesis relative to s in this case.

Inductive step $\beta - 1 \to \beta$. Suppose that the inclusion $\mathcal{V}_{s,\beta-1} \subset \mathcal{W}_s$ is true for some $\beta \in \mathbb{Z}_{\geqslant 0}$ and let us prove that the inclusion $\mathcal{V}_{s,\beta} \subset \mathcal{W}_s$ is true. It is clearly sufficient to prove that any $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ with $(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in (\{0,\ldots,\beta\} \times \mathbb{Z}_{\geqslant 0}^{s-1}) \times (\overline{\mathbb{Q}}^{\times})^{s} \times \mathbb{Q}_{>0}^{s}$ belongs to $\mathcal{W}_s$. Consider such a triple $(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})$.

Let us first note that, for all $i \in \mathbb{Z}_{\geqslant 0}$, we have

$$\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^{i}\boldsymbol{a}} - c\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^{i+1}\boldsymbol{a}} \in \mathcal{V}_{s,\beta-1}$$

with $c = (\lambda_1 \cdots \lambda_s)^{-1}$; this follows immediately from Lemma 21 after noticing that $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^{i+1}\boldsymbol{a}}(z) = \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^{i}\boldsymbol{a}}(z^p)$. This implies that, for all $u \in \mathbb{Z}_{\geqslant 1}$,

$$\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} - c^u \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^u\boldsymbol{a}} = \sum_{i=0}^{u-1} c^i (\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^i\boldsymbol{a}} - c\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^{i+1}\boldsymbol{a}}) \in \mathcal{V}_{s,\beta-1}. \tag{57}$$

Since $\mathcal{V}_{s,\beta-1} \subset \mathcal{W}_{s-1}$ by induction, (57) shows that, in order to prove that $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ belongs to $\mathcal{W}_s$, it is equivalent to prove that $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^u\boldsymbol{a}}$ belongs to $\mathcal{W}_s$ for some $u \in \mathbb{Z}_{\geqslant 0}$. So, up to replacing $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ by $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p^u\boldsymbol{a}}$ for $u \in \mathbb{Z}_{\geqslant 0}$ large enough, we can and will assume that the denominators of the entries of $\boldsymbol{a}$ are relatively prime with p.

Now, for any $\boldsymbol{e} = (e_1, \ldots, e_s) \in \mathbb{Z}^s_{\geqslant 0}$, we set $\boldsymbol{a_e} = \left(\frac{a_1}{p^{e_1}}, \ldots, \frac{a_s}{p^{e_s}}\right)$. We claim that

$$\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \operatorname{Span}_{\overline{\mathbb{Q}}} \{\xi_{\boldsymbol{\alpha}',\boldsymbol{\lambda},\boldsymbol{a_e}} \mid \boldsymbol{\alpha}' \in \mathbb{Z}^s_{\geqslant 0}\} + \mathcal{V}_{s-1}. \tag{58}$$

It is sufficient to prove this claim when $\boldsymbol{e}$ is of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$: the general case is obtained by applying these special cases iteratively. Let us explain the proof of (58) in the case when $s = 2$, the general case being similar but requiring unpleasant notations. We split our study in different cases.

Case $\boldsymbol{e} = (1, 0)$ and $\alpha_2 \neq 0$. We have

$$\begin{aligned}
\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} &= \sum_{k_1,k_2 \geqslant 1} k_1^{\alpha_1} k_2^{\alpha_2} \lambda_1^{k_1} \lambda_2^{k_1+k_2} z^{-\frac{a_1}{p^{k_1}} - \frac{a_2}{p^{k_1+k_2}}} \\
&= \sum_{k_1 \geqslant 0, k_2 \geqslant 1} (k_1+1)^{\alpha_1} k_2^{\alpha_2} \lambda_1^{k_1+1} \lambda_2^{k_1+k_2+1} z^{-\frac{a_1/p}{p^{k_1}} - \frac{a_2}{p^{k_1+k_2+1}}} \\
&= \sum_{k_1 \geqslant 0, k_2 \geqslant 2} \lambda_1 (k_1+1)^{\alpha_1} (k_2-1)^{\alpha_2} \lambda_1^{k_1} \lambda_2^{k_1+k_2} z^{-\frac{a_1/p}{p^{k_1}} - \frac{a_2}{p^{k_1+k_2}}} \\
&= \sum_{k_1 \geqslant 0, k_2 \geqslant 1} \lambda_1 (k_1+1)^{\alpha_1} (k_2-1)^{\alpha_2} \lambda_1^{k_1} \lambda_2^{k_1+k_2} z^{-\frac{a_1/p}{p^{k_1}} - \frac{a_2}{p^{k_1+k_2}}} \\
&= \sum_{k_1 \geqslant 0, k_2 \geqslant 1} \lambda_1 \sum_{j=0}^{\alpha_1} \sum_{j'=0}^{\alpha_2} \binom{\alpha_1}{j} \binom{\alpha_2}{j'} (-1)^{\alpha_2 - j'} k_1^j k_2^{j'} \lambda_1^{k_1} \lambda_2^{k_1+k_2} z^{-\frac{a_1/p}{p^{k_1}} - \frac{a_2}{p^{k_1+k_2}}}.
\end{aligned}$$

Using the decomposition

$$\sum_{k_1 \geqslant 0, k_2 \geqslant 1} = \sum_{k_1 \geqslant 1, k_2 \geqslant 1} + \sum_{k_1 = 0, k_2 \geqslant 1},$$

we get

$$\begin{aligned}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} &= \sum_{j=0}^{\alpha_1}\sum_{j'=0}^{\alpha_2}\lambda_1\binom{\alpha_1}{j}\binom{\alpha_2}{j'}(-1)^{\alpha_2-j'}\xi_{(j,j'),\boldsymbol{\lambda},\boldsymbol{a_e}}\\ &\quad+\sum_{j'=0}^{\alpha_2}\lambda_1\binom{\alpha_2}{j'}(-1)^{\alpha_2-j'}z^{-\frac{a_1}{p}}\xi_{(j'),(\lambda_2),(a_2)}.\end{aligned}$$

This proves (58) in the case $\boldsymbol{e}=(1,0)$ and $\alpha_2\neq 0$.

Case $\boldsymbol{e}=(1,0)$ and $\alpha_2=0$. We have

$$\begin{aligned}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} &= \sum_{k_1,k_2\geqslant 1} k_1^{\alpha_1}\lambda_1^{k_1}\lambda_2^{k_1+k_2}z^{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}}\\ &= \sum_{k_1\geqslant 0,k_2\geqslant 1}(k_1+1)^{\alpha_1}\lambda_1^{k_1+1}\lambda_2^{k_1+k_2+1}z^{-\frac{a_1/p}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2+1}}}\\ &= \sum_{k_1\geqslant 0,k_2\geqslant 2}\lambda_1(k_1+1)^{\alpha_1}\lambda_1^{k_1}\lambda_2^{k_1+k_2}z^{-\frac{a_1/p}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}}\\ &= \sum_{k_1\geqslant 0,k_2\geqslant 2}\lambda_1\sum_{j=0}^{\alpha_1}\binom{\alpha_1}{j}k_1^j\lambda_1^{k_1}\lambda_2^{k_1+k_2}z^{-\frac{a_1/p}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}}.\end{aligned}$$

Using the decomposition

$$\sum_{k_1\geqslant 0,k_2\geqslant 2}=\sum_{k_1\geqslant 1,k_2\geqslant 1}+\sum_{k_1=0,k_2\geqslant 1}-\sum_{k_1\geqslant 1,k_2=1}-\sum_{k_1=0,k_2=1},$$

we get

$$\begin{aligned}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} &= \sum_{j=0}^{\alpha_1}\lambda_1\binom{\alpha_1}{j}\xi_{(j,0),\boldsymbol{\lambda},\boldsymbol{a_e}}+\lambda_1 z^{-\frac{a_1}{p}}\xi_{(0),(\lambda_2),(a_2)}\\ &\quad-\sum_{j=0}^{\alpha_1}\lambda_1\lambda_2\binom{\alpha_1}{j}\xi_{(j),(\lambda_1\lambda_2),(\frac{a_1+a_2}{p})}-\lambda_1\lambda_2 z^{-\frac{a_1+a_2}{p}}.\end{aligned}$$

This proves (58) when $\boldsymbol{e}=(1,0)$ in the case $\alpha_2=0$ as well.

Case $\boldsymbol{e}=(0,1)$. We have

$$\begin{aligned}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} &= \sum_{k_1,k_2\geqslant 1} k_1^{\alpha_1}k_2^{\alpha_2}\lambda_1^{k_1}\lambda_2^{k_1+k_2}z^{-\frac{a_1}{p^{k_1}}-\frac{a_2}{p^{k_1+k_2}}}\\ &= \sum_{k_1\geqslant 1,k_2\geqslant 0}k_1^{\alpha_1}(k_2+1)^{\alpha_2}\lambda_1^{k_1}\lambda_2^{k_1+k_2+1}z^{-\frac{a_1}{p^{k_1}}-\frac{a_2/p}{p^{k_1+k_2}}}\\ &= \sum_{j=0}^{\alpha_2}\lambda_2\binom{\alpha_2}{j}\xi_{(\alpha_1,j),\boldsymbol{\lambda},\boldsymbol{a_e}}+\lambda_2\xi_{(\alpha_1),(\lambda_1),(a_1+a_2/p)}.\end{aligned}$$

This proves (58) when $\boldsymbol{e}=(0,1)$.

Applying (58) with $\boldsymbol{e} = (e_1, \dots, e_s)$ where $e_i \in \mathbb{Z}_{\geqslant 0}$ is such that p^e is the greatest power of p dividing the numerator of a_i, so that $\boldsymbol{a_e} \in \mathbb{N}^s_{(p)}$, and using the fact that $\mathcal{V}_{s-1} \subset \mathcal{W}_{s-1}$ by induction, we get

$$\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \mathcal{W}_s + \mathcal{V}_{s-1} \subset \mathcal{W}_s + \mathcal{W}_{s-1} \subset \mathcal{W}_s.$$

This concludes the inductive step and the demonstration. □

5.2. **Proof of Proposition 10.** Fix $r \in \{1, 2, 3\}$. In order to prove Proposition 10, we have to prove that if a generalized p-Mahler series f admits a decomposition of the form (3)-(4) such that all the Puiseux series $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ satisfy $(\mathcal{O}_r)$, then the Puiseux series involved in its standard decomposition satisfy $(\mathcal{O}_r)$ as well. In order to prove this, note that Lemma 41 implies that the Puiseux series involved in the standard decomposition of f are $\overline{\mathbb{Q}}[z^{-\frac{1}{*}}]$-linear combinations of the $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$. Since the set of p-Mahler Puiseux series satisfying $(\mathcal{O}_r)$ is a $\overline{\mathbb{Q}}[z^{-\frac{1}{*}}]$-module, we get that the Puiseux series involved in the standard decomposition of f satisfy $(\mathcal{O}_r)$ as well. This concludes the proof.

6. Purity Theorem: proof of Theorem 11

Theorem 11 is proved in Section 6.5 below. The proof uses a result from [ABS23] that we shall first remind.

6.1. **Reminders on the p-Mahler denominator.** Theorem 8 ensures that any p-Mahler Laurent series $f \in \overline{\mathbb{Q}}((z))$ satisfies the growth condition $(\mathcal{O}_1)$. According to [ABS23], one can determine whether or not it satisfies one of the stronger conditions $(\mathcal{O}_2)$ or $(\mathcal{O}_3)$ by looking at its p-Mahler denominator. Let us briefly remind this.

Recall that we let ϕ_p denote the operator which maps any $f \in \mathscr{H}$ to $f(z^p)$. So, for any $f \in \mathscr{H}$ and $i \in \mathbb{Z}_{\geqslant 0}$, $\phi_p^i(f) = f(z^{p^i})$.

Definition 42. *The p-Mahler denominator $\mathfrak{d}_f$ of a p-Mahler Laurent series $f \in \overline{\mathbb{Q}}((z))$ is the monic generator of the ideal of $\overline{\mathbb{Q}}[z]$ given by*

$$(59)\qquad \left\{ P \in \overline{\mathbb{Q}}[z] \;\middle|\; Pf \in \sum_{i=1}^{d} \overline{\mathbb{Q}}[z]\phi_p^i(f) \text{ for some } d \in \mathbb{Z}_{\geqslant 1} \right\}.$$

Remark 43. *1) In [ABS23], the previous definition is formulated for p-Mahler series $f \in \overline{\mathbb{Q}}[[z]]$. Its extension to $\overline{\mathbb{Q}}((z))$ is straightforward.*

2) The p-Mahler equations considered in [ABS23] have coefficients in $\overline{\mathbb{Q}}(z)$, whereas the equations considered in the present paper have coefficients in the bigger field $\mathbb{K}_\infty$. Note that $f \in \overline{\mathbb{Q}}((z))$ satisfies a p-Mahler equation with coefficients in $\overline{\mathbb{Q}}(z)$ if and only if it satisfies a p-Mahler equation with coefficients in $\mathbb{K}_\infty$. Indeed, assume that $f \in \overline{\mathbb{Q}}[[z]]$ satisfies a p-Mahler equation of the form (1) with $a_0, \dots, a_d \in \overline{\mathbb{Q}}(z^{\frac{1}{m}})$ for some $m \in \mathbb{Z}_{\geqslant 1}$. Without *loss of generality, we can assume that* $a_0 = 1$. *Let* G *be the Galois group of the finite Galois extension* $\overline{\mathbb{Q}}((z^{\frac{1}{m}}))$ *of* $\overline{\mathbb{Q}}((z))$. *Then,* f *satisfies*

$$b_0 f + b_1\phi_p(f) + \cdots + b_d\phi_p^d(f) = 0$$

with $b_i = \frac{1}{|G|}\sum_{\sigma\in G}\sigma(a_i) \in \overline{\mathbb{Q}}(z)$ *and this equation is non trivial because* $b_0 = 1$.

In what follows, we denote by $\mathcal{U}$ the set of complex roots of unity and by $\mathcal{U}_p$ the set of roots of unity whose order is not coprime with p. We recall the following result.

Theorem 44 ([ABS23, Theorems 6.1 & 7.1])**.** *Consider a* p*-Mahler Laurent series* $f \in \overline{\mathbb{Q}}((z))$. *We have the following:*

- f *satisfies* $(\mathcal{O}_2)$ *if and only if every non-zero root of the* p*-Mahler denominator of* f *belong to* $\mathcal{U}$*;*
- f *satisfies* $(\mathcal{O}_3)$ *if and only if every non-zero root of the* p*-Mahler denominator of* f *belong to* $\mathcal{U}_p$.

Remark 45. *In* [ABS23]*, the previous result is stated for any* p*-Mahler series* $f \in \overline{\mathbb{Q}}[[z]]$. *Theorem 44 follows from* [ABS23] *by using the following remark. Consider* $g = z^\nu f$ *with* $\nu \in \mathbb{Z}_{\geqslant 0}$ *large enough so that* $g \in \overline{\mathbb{Q}}[[z]]$. *Then,* f *satisfies* $(\mathcal{O}_2)$ *(resp.* $(\mathcal{O}_3)$*) if and only if* g *satisfies the same property. Moreover, every non-zero root of the* p*-Mahler denominator of* f *belong to* $\mathcal{U}_p$ *(resp.* $\mathcal{U}$*) if and only if the* p*-Mahler denominator of* g *satisfies the same property. Now, Theorem 44 follows from* [ABS23, Theorems 6.1 & 7.1] *applied to the* p*-Mahler series* g.

6.2. p-Mahler denominator for generalized p-Mahler series.

We extend the definition of p-Mahler denominator to generalized p-Mahler series in the following obvious way.

Definition 46. *The* p*-Mahler denominator* $\mathfrak{d}_f$ *of a generalized* p*-Mahler series* $f \in \mathcal{R}$ *is the monic generator of the ideal of* $\overline{\mathbb{Q}}[z]$ *given by*

$$\left\{P \in \overline{\mathbb{Q}}[z] \;\Big|\; Pf \in \sum_{i=1}^{d}\overline{\mathbb{Q}}[z]\phi_p^i(f) \text{ for some } d \in \mathbb{Z}_{\geqslant 1}\right\} \tag{60}$$

if the latter ideal is non trivial; otherwise, we set $\mathfrak{d}_f = 0$.

Remark 47. *1) Contrary to the case when* $f \in \overline{\mathbb{Q}}((z))$ *considered in Section 6.1, the ideal* (60) *may be trivial and, hence, the* p*-Mahler denominator* $\mathfrak{d}_f$ *may be equal to* 0. *For instance, this is the case for* $f(z) = z^{\frac{1}{p}}$.

2) One can prove that, for any generalized p*-Mahler series* f *having a decomposition of the form* (3)-(4) *such that the* $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ *belong to* $\overline{\mathbb{Q}}((z))$, *the ideal* (60) *is non trivial. This can be proved using arguments used to prove Proposition 51. As this will not be used in this paper, we do not include the details.*

6.3. **First step toward the proof of Theorem 11: denominator and growth class of a basis of solutions.** A first step toward the proof of Theorem 11 is to prove that the zero locus of the coefficient a_0 of a p-Mahler equation provides informations on the class $(\mathscr{P}-\mathbb{O}_r)$ of its generalized p-Mahler series solutions.

Proposition 48. *Consider a p-Mahler equation*

$$a_d\phi_p^d(y)+a_{d-1}\phi_p^{d-1}(y)+\cdots+a_0y=0 \tag{61}$$

with $a_0,\ldots,a_d\in\overline{\mathbb{Q}}[z]$ such that $a_0a_d\neq 0$. This equation has d $\overline{\mathbb{Q}}$-linearly independent generalized p-Mahler series solutions satisfying

- *$(\mathscr{P}-\mathbb{O}_1)$ in any case;*
- *$(\mathscr{P}-\mathbb{O}_2)$ if the non-zero roots of a_0 belong to $\mathcal{U}$;*
- *$(\mathscr{P}-\mathbb{O}_3)$ if the non-zero roots of a_0 belong to $\mathcal{U}_p$.*

Proposition 48 is proved below, after the following lemmas.

Lemma 49. *Let f,g be generalized p-Mahler series such that*

$$\mathfrak{e}g=\sum_{i=1}^{n}a_i\phi_p^i(g)+f \tag{62}$$

for some $\mathfrak{e}\in\overline{\mathbb{Q}}[z]\backslash\{0\}$ and some $a_1,\ldots,a_n\in\overline{\mathbb{Q}}[z]$. Then, the p-Mahler denominator $\mathfrak{d}_g$ of g divides $\mathfrak{e}\mathfrak{d}_f$ where $\mathfrak{d}_f$ is the p-Mahler denominator of f.

Proof. If $\mathfrak{d}_f=0$, there is nothing to prove since any element of $\overline{\mathbb{Q}}[z]$ divides $\mathfrak{e}\mathfrak{d}_f=0$.

We now assume that $\mathfrak{d}_f\neq 0$. We have

$$\mathfrak{d}_f f=\sum_{j=1}^{m}b_j\phi_p^j(f)$$

for some $b_1,\ldots,b_m\in\overline{\mathbb{Q}}[z]$. It follows that

$$\mathfrak{e}\mathfrak{d}_f g=\sum_{i=1}^{n}\mathfrak{d}_f a_i\phi_p^i(g)+\sum_{j=1}^{m}b_j\phi_p^j(f). \tag{63}$$

But, applying ϕ_p^j to (62) with $j\in\{1,\ldots,m\}$, we get

$$\phi_p^j(f)=\phi_p^j(\mathfrak{e})\phi_p^j(g)-\sum_{i=1}^{n}\phi_p^j(a_i)\phi_p^{i+j}(g). \tag{64}$$

Substituting the $\phi_p^j(f)$ in (63) with the right-hand side of (64) we obtain

$$\mathfrak{e}\mathfrak{d}_f g\in\sum_{i=1}^{m+n}\overline{\mathbb{Q}}[z]\phi_p^i(g).$$

Thus, $\mathfrak{e}\mathfrak{d}_f$ belongs to the ideal (60). Then, $\mathfrak{d}_g$ divides $\mathfrak{e}\mathfrak{d}_f$, whence the result. □

Lemma 50. *Let $f, g \in \overline{\mathbb{Q}}((z))$ be p-Mahler Laurent series such that*

$$\mathfrak{e} g = \sum_{i=1}^{n} a_i \phi_p^i(g) + f \tag{65}$$

for some $\mathfrak{e} \in \overline{\mathbb{Q}}[z] \backslash \{0\}$ and some $a_1, \ldots, a_n \in \overline{\mathbb{Q}}[z]$. Then, g satisfies

- $(\mathcal{O}_1)$ *in any case;*
- $(\mathcal{O}_2)$ *if f satisfies $(\mathcal{O}_2)$ and if the non-zero roots of $\mathfrak{e}$ belong to $\mathcal{U}$;*
- $(\mathcal{O}_3)$ *if f satisfies $(\mathcal{O}_3)$ and if the non-zero roots of $\mathfrak{e}$ belong to $\mathcal{U}_p$.*

Proof. The first assertion follows immediately from Theorem 8. In order to prove the last two assertions, we first remind that Lemma 49 ensures that the p-Mahler denominator of g divides $\mathfrak{e}\mathfrak{d}_f$ where $\mathfrak{d}_f$ is the p-Mahler denominator of f. Then, the result follows immediately from Theorem 44. □

Proof of Proposition 48. Let $\phi_p(Y) = AY$ be the p-Mahler system associated to the p-Mahler equation (61), where

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -\frac{a_0}{a_d} & -\frac{a_1}{a_d} & & & -\frac{a_{d-1}}{a_d} \end{pmatrix}. \tag{66}$$

The entries of the first line of any fundamental matrix of solutions of this system constitute a full basis of solutions of (61). Thus, in order to prove that (61) has d $\overline{\mathbb{Q}}$-linearly independent generalized p-Mahler series solutions satisfying $(\mathcal{O}_r)$, it is sufficient to prove that $\phi_p(Y) = AY$ has a fundamental matrix of solutions whose entries are generalized p-Mahler series satisfying $(\mathscr{P}{-}\mathcal{O}_r)$. By Theorem 27 and Remark 28, the system (44) has a fundamental matrix of solutions of the form

$$F_1 F_2 e_C$$

where

- $F_1 \in \mathrm{GL}_d(\mathscr{P})$ is such that
$$\phi_p(F_1)\Theta = AF_1 \tag{67}$$
for some upper triangular matrix Θ with diagonal coefficients in $\overline{\mathbb{Q}}^{\times}$ and with upper-diagonal coefficients in $\bigcup_{\gamma \in \mathbb{Q}_{>0}} \overline{\mathbb{Q}}[z^{-\gamma}]$;
- F_2 has entries in $\mathcal{V}$;
- the entries of e_C are $\overline{\mathbb{Q}}$-linear combinations of some $e_c \ell^j$.

In order to prove that $\phi_p(Y) = AY$ has a fundamental matrix of solutions whose entries are generalized p-Mahler series satisfying $(\mathcal{O}_r)$, it is sufficient to prove that the entries of F_1 satisfy $(\mathcal{O}_r)$. In order to conclude the proof, it is thus sufficient to prove that any of the entries of F_1 satisfy the following property, that we denote by $(\diamond)$:

- $(\mathcal{O}_1)$ in any case;
- $(\mathcal{O}_2)$ if the non-zero roots of a_0 belong to $\mathcal{U}$;
- $(\mathcal{O}_3)$ if the non-zero roots of a_0 belong to $\mathcal{U}_p$.

We will deduce this from equation (67). Note that Property $(\diamond)$ is invariant under sums, product by an element of $\overline{\mathbb{Q}}[z^{-1}]$ and under ϕ_p. We first notice that, for any $k \in \mathbb{Z}_{\geqslant 1}$, the entries of F_1 satisfy $(\mathcal{O}_r)$ if and only if the entries of $F_1(z^k)$ satisfy $(\mathcal{O}_r)$. Thus, up to replacing z by z^k, $F_1(z)$ by $F_1(z^k)$, $\Theta(z)$ by $\Theta(z^k)$ and $A(z)$ by $A(z^k)$ in (67) for a suitable $k \in \mathbb{Z}_{\geqslant 1}$, we can and will assume that F_1 has coefficients in $\overline{\mathbb{Q}}((z))$ and that Θ has coefficients in $\overline{\mathbb{Q}}[z^{-1}]$ (note that the fact that the non-zero roots of a_0 belong to $\mathcal{U}$ (resp. $\mathcal{U}_p$) implies that the non-zero roots of $a_0(z^k)$ belong to $\mathcal{U}$ (resp. $\mathcal{U}_p$) as well). Equation (67) can be rewritten as follows:

$$a_0 F_1 \Lambda = B\phi_p(F_1) \tag{68}$$

where

$$\Lambda = \Theta^{-1} = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix}$$

has diagonal coefficients $\lambda_1, \ldots, \lambda_d \in \overline{\mathbb{Q}}^{\times}$ and upper-diagonal coefficients in $\overline{\mathbb{Q}}[z^{-1}]$ and where

$$B = A^{-1} = \begin{pmatrix} -a_1 & \cdots & \cdots & -a_d \\ a_0 & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ & & a_0 & 0 \end{pmatrix}.$$

Setting $F_1 = (f_{i,j})_{1 \leqslant i,j \leqslant d}$, we deduce from (68) that, for all $i, j \in \{1, \ldots, d\}$,

$$a_0 \left(\sum_{l=1}^{j-1} * f_{i,l} + \lambda_j f_{i,j} \right) = \begin{cases} \sum_{k=1}^{d} -a_k \phi_p(f_{k,j}) & \text{if } i = 1, \\ a_0 \phi_p(f_{i-1,j}) & \text{if } i \in \{2, \ldots, d\} \end{cases}$$

where the symbol $*$ stands for elements of $\overline{\mathbb{Q}}[z^{-1}]$.

For $j = 1$, this gives

$$a_0 \lambda_1 f_{i,1} = \begin{cases} \sum_{k=1}^{d} -a_k \phi_p(f_{k,1}) & \text{if } i = 1, \\ a_0 \phi_p(f_{i-1,1}) & \text{if } i \in \{2, \ldots, d\} \end{cases}.$$

This implies that, for $i \in \{2, \ldots, d\}$, $f_{i,1} = \lambda_1^{-(i-1)} \phi_p^{i-1}(f_{1,1})$ and, hence,

$$a_0 \lambda_1 f_{1,1} = \sum_{k=1}^{d} -a_k \lambda_1^{-(k-1)} \phi_p^{k}(f_{1,1}).$$

Thus, the p-Mahler denominator of $f_{1,1}$ divides a_0 and it follows from Theorem 44 that $f_{1,1}$ satisfies $(\diamond)$. Therefore, for any $i \in \{2, \ldots, d\}$, $f_{i,1} = \lambda_1^{-(i-1)} \phi_p^{i-1}(f_{1,1})$ satisfies $(\diamond)$ as well.

For $j = 2$,

$$a_0 \left(* f_{i,1} + \lambda_2 f_{i,2} \right) = \begin{cases} \sum_{k=0}^{d} -a_k \phi_p(f_{k,2}) & \text{if } i = 1, \\ a_0 \phi_p(f_{i-1,2}) & \text{if } i \in \{2, \ldots, d\} \end{cases}$$

where the symbol $*$ stands for an element of $\overline{\mathbb{Q}}[z^{-1}]$. This implies that, for $i \in \{2, \ldots, d\}$, $f_{i,2} = \lambda_2^{-(i-1)} \phi_p^{i-1}(f_{1,2}) + \star$ and, hence

$$a_0 \lambda_2 f_{1,2} = \sum_{k=1}^{d} -a_k \lambda_2^{-(k-1)} \phi_p^{k}(f_{1,2}) + \star$$

where the symbol $\star$ stands for p-Mahler elements of $\overline{\mathbb{Q}}((z))$ satisfying $(\diamond)$. It follows from Lemma 50 that $f_{1,2}$ satisfies $(\diamond)$. Therefore, for any $i \in \{2, \ldots, d\}$, $f_{i,2} = \lambda_2^{-(i-1)} \phi_p^{i-1}(f_{1,2}) + \star$ satisfies $(\diamond)$ as well.

Iterating this argument, we find that, for all $i, j \in \{1, \ldots, d\}$, $f_{i,j}$ satisfies $(\diamond)$. □

6.4. **Second step toward the proof of Theorem 11: roots of the p-Mahler denominator of generalized p-Mahler series.** The next step toward the proof of Theorem 11 is the following result.

Proposition 51. *Let f be a generalized p-Mahler series of the form*

$$f = \sum_{(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}, (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in \boldsymbol{\Lambda}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j$$

where the $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ belong to $\overline{\mathbb{Q}}((z))$. We assume that the vectors $\boldsymbol{a}$ involved in the support of the sum have entries in $\mathbb{Z}_{>0}$. Then, the following hold:

- *if f satisfies $(\mathscr{P} - \mathscr{O}_2)$ then the p-Mahler denominator of f has all its non-zero roots in $\mathcal{U}$;*
- *if f satisfies $(\mathscr{P} - \mathscr{O}_3)$ then the p-Mahler denominator of f has all its non-zero roots in $\mathcal{U}_p$.*

In particular, in both cases, the p-Mahler denominator of f is non-zero.

Proposition 51 is proved at the end of this subsection, after the following lemma.

Lemma 52. *Proposition 51 holds if $f \in \mathscr{H}$.*

Proof. Let us first introduce some notations.

We let M be the set of p-Mahler Laurent series $h \in \overline{\mathbb{Q}}((z))$ satisfying $(\mathscr{O}_2)$ (resp. $(\mathscr{O}_3)$). It is a $\overline{\mathbb{Q}}[z, z^{-1}]$-module invariant by ϕ_p.

For any $s \in \mathbb{Z}_{\geqslant 0}$, we let $\mathscr{W}_s$ be the set of Hahn series of the form

$$\sum_{(\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in \bigcup_{t \in \{0, \ldots, s\}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^{\times})^t \times \mathbb{Z}_{>0}^t} f_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$$

where the sum has finite support and where the $f_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ belong M. In particular, $\mathscr{W}_0 = M$.

For any $s \in \mathbb{Z}_{\geqslant 0}$, we consider the filtration $(\mathscr{W}_{s,\beta})_{\beta \in \mathbb{Z}_{\geqslant 0}}$ of $\mathscr{W}_s$ defined as follows: $\mathscr{W}_{s,\beta}$ is the set of Hahn series of the form

$$\sum_{\boldsymbol{\alpha} \in \{0,\dots,\beta\} \times \mathbb{Z}_{\geqslant 0}^{s-1},\, \boldsymbol{\lambda} \in (\overline{\mathbb{Q}}^{\times})^s,\, \boldsymbol{a} \in \mathbb{Z}_{>0}^s} f_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} + \sum_{(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \bigcup_{t \in \{0,\dots,s-1\}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^{\times})^t \times \mathbb{Z}_{>0}^t} f_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \tag{69}$$

where the sums have finite support and where the $f_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ belong to M. In particular, $\mathscr{W}_{0,\beta} = \mathscr{W}_0 = M$.

It will be convenient to set, for any $s \in \mathbb{Z}_{\geqslant 1}$, $\mathscr{W}_{s,-1} = \mathscr{W}_{s-1}$.

Since M is a $\overline{\mathbb{Q}}[z, z^{-1}]$-module, so are $\mathscr{W}_s$ and $\mathscr{W}_{s,\beta}$. Moreover, Lemma 21 implies that the $\overline{\mathbb{Q}}[z, z^{-1}]$-modules $\mathscr{W}_s$ and $\mathscr{W}_{s,\beta}$ are invariant by ϕ_p.

Since any Hahn series satisfying the assumptions of Proposition 51 belongs to $\mathscr{W}_s$ for some $s \in \mathbb{Z}_{\geqslant 0}$, the statement of the present lemma can be restated as follows: Proposition 51 holds for any element f of $\mathscr{W}_s$, for any $s \in \mathbb{Z}_{\geqslant 0}$. In order to prove this, we argue by induction on s.

Base case $s = 0$. If $s = 0$, then $\mathscr{W}_s = \mathscr{W}_0 = M$ and the result follows immediately from Theorem 44.

Inductive step $s - 1 \to s$. Suppose that the result is true for the elements of $\mathscr{W}_{s-1}$ for some $s \in \mathbb{Z}_{\geqslant 1}$ and let us prove that the result is true for the elements of $\mathscr{W}_s$. Since $\mathscr{W}_s = \bigcup_{\beta \in \mathbb{Z}_{\geqslant -1}} \mathscr{W}_{s,\beta}$, it is equivalent to prove that the result is true for any element of $\mathscr{W}_{s,\beta}$ for any $\beta \in \mathbb{Z}_{\geqslant -1}$. In order to prove this, we argue by induction on β.

Base case $\beta = -1$. If $\beta = -1$, then $\mathscr{W}_{s,\beta} = \mathscr{W}_{s,-1} = \mathscr{W}_{s-1}$ and the result follows from the inductive hypothesis relative to s.

Inductive step $\beta - 1 \to \beta$. Suppose that the result is true for the elements of $\mathscr{W}_{s,\beta-1}$ for some $\beta \in \mathbb{Z}_{\geqslant 0}$ and let us prove that the result is true for the elements of $\mathscr{W}_{s,\beta}$. Any element f of $\mathscr{W}_{s,\beta}$ can be written as follows

$$f = \sum_{\boldsymbol{\alpha} \in \{\beta\} \times \mathbb{Z}_{\geqslant 0}^{s-1}, \boldsymbol{\lambda} \in (\overline{\mathbb{Q}}^{\times})^s, \boldsymbol{a} \in \mathbb{Z}_{>0}^s} f_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \mod \mathscr{W}_{s,\beta-1}. \tag{70}$$

We will say that the (s, β)-length of such an f is at most $l \in \mathbb{Z}_{\geqslant 0}$ if the number of terms in the above sum is at most l. We now argue by induction on l.

Base case $l = 0$. If $l = 0$, then f belongs to $\mathscr{W}_{s,\beta-1}$ and the result follows by the inductive hypothesis relative to β.

Inductive step $l - 1 \to l$. We assume that the result is true for any element of $\mathscr{W}_{s,\beta}$ of (s, β)-length at most $l - 1$ and we will prove that it is true for any element of $\mathscr{W}_{s,\beta}$ of (s, β)-length at most l. So, we consider an element f of $\mathscr{W}_{s,\beta}$ of (s, β)-length at most l and we consider its decomposition (70). If all the $f_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ involved in this decomposition are 0, then we are in the base case $l = 0$ that has already been treated. So, we can assume that there is an index $(\boldsymbol{\alpha}^0, \boldsymbol{\lambda}^0, \boldsymbol{a}^0)$ such that $f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0} \neq 0$. Since $f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}$ satisfies $(\mathcal{O}_2)$ (resp. $(\mathcal{O}_3)$), it follows from Theorem 44 that the non-zero roots of the p-Mahler denominator $\mathfrak{d} = \mathfrak{d}_{f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}}$ of $f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}$ belong to $\mathcal{U}$ (resp. to $\mathcal{U}_p$). By definition of the p-Mahler denominator, there exist $a_1, \ldots, a_d \in \overline{\mathbb{Q}}[z]$ such that

$$\mathfrak{d} f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0} = \sum_{j=1}^{d} a_j \phi_p^j(f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0})\,.$$

We can write

$$f = h_0 + f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}\xi_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}$$

with $h_0 \in \mathcal{W}_{s,\beta}$ of (s,β)-length at most $l-1$ and we have

$$\begin{aligned}\mathfrak{d} f &= \mathfrak{d} h_0 + \mathfrak{d} f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}\xi_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}\\ &= \mathfrak{d} h_0 + \sum_{j=1}^{d} a_j \phi_p^j(f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0})\xi_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}.\end{aligned}$$

But, it follows from Lemma 21 that

$$\phi_p^j(f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0})\xi_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0} = c^j \phi_p^j(f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}\xi_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}) \mod \mathcal{W}_{s,\beta-1}.$$

where $c \in \overline{\mathbb{Q}}^\times$ is the inverse of the product of the coordinates of $\boldsymbol{\lambda}^0$. So, we get the following equalities modulo $\mathcal{W}_{s,\beta-1}$:

$$\begin{aligned}\mathfrak{d} f &\equiv \mathfrak{d} h_0 + \sum_{j=1}^{d} a_j c^j \phi_p^j(f_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0}\xi_{\boldsymbol{\alpha}^0,\boldsymbol{\lambda}^0,\boldsymbol{a}^0})\\ &\equiv \mathfrak{d} h_0 - \sum_{j=1}^{d} a_j c^j \phi_p^j(h_0) + \sum_{i=1}^{d} a_j c^j \phi_p^j(f).\end{aligned}$$

Hence, we have

$$h := \mathfrak{d} f - \sum_{j=1}^{d} a_j c^j \phi_p^j(f) \equiv \mathfrak{d} h_0 - \sum_{i=1}^{d} a_i c^i \phi_p^j(h_0) \mod \mathcal{W}_{s,\beta-1}$$

Using Lemma 21 again and (71), we see that h is – as h_0 – an element of $\mathcal{W}_{s,\beta}$ of (s,β)-length at most $l-1$. By the induction hypothesis on the (s,β)-length, the p-Mahler denominator $\mathfrak{d}_h$ of h has its non-zero roots in $\mathcal{U}$ (resp. in $\mathcal{U}_p$). Then, using (71) and Lemma 49, we get that the p-Mahler denominator $\mathfrak{d}_f$ of f divides $\mathfrak{d} \cdot \mathfrak{d}_h$ and, hence, has its non-zero roots in $\mathcal{U}$ (resp. in $\mathcal{U}_p$). This concludes the proof. □

Proof of Proposition 51. To any generalized p-Mahler series of the form

$$f = \sum_{(c,j)\in\overline{\mathbb{Q}}^\times\times\mathbb{Z}_{\geqslant 0},(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in\bigcup_{t\in\mathbb{Z}_{\geqslant 0}}\mathbb{Z}_{\geqslant 0}^t\times(\overline{\mathbb{Q}}^\times)^t\times\mathbb{Z}_{>0}^t} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j \tag{71}$$

where the $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \overline{\mathbb{Q}}((z))$ are p-Mahler Laurent series, we attach a number $K(f) \in \mathbb{Z}_{\geqslant 1}$ defined as follows. For any $(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}$, we set

$$h_{c,j} = \sum_{(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}) \in \bigcup_{t \in \mathbb{Z}_{\geqslant 0}} \mathbb{Z}^t_{\geqslant 0} \times (\overline{\mathbb{Q}}^{\times})^t \times \mathbb{Z}^t_{>0}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \mathscr{H}, \tag{72}$$

so that

$$f = \sum_{(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}} h_{c,j} e_c \ell^j. \tag{73}$$

We let $\mathcal{C}(f)$ be the set of $c \in \overline{\mathbb{Q}}^{\times}$ such that $h_{c,j} \neq 0$ for some $j \in \mathbb{Z}_{\geqslant 0}$. For each $c \in \mathcal{C}(f)$, let $J(f,c)$ denote the maximal $j \in \mathbb{Z}_{\geqslant 0}$ such that $h_{c,j} \neq 0$. We set $K(f) = \sum_{c \in \mathcal{C}(f)} (1 + J(f,c))$.

In order to prove that Proposition 51 holds true for any generalized p-Mahler series f of the form (71) satisfying $(\mathscr{P} - \mathcal{O}_2)$ (resp. $(\mathscr{P} - \mathcal{O}_3)$), it is of course equivalent to prove that it is is true for any generalized p-Mahler series f of the form (71) satisfying $(\mathscr{P} - \mathcal{O}_2)$ (resp. $(\mathscr{P} - \mathcal{O}_3)$) such that $K(f) \leqslant k$ for some $k \in \mathbb{Z}_{\geqslant 1}$. Let us prove this by induction on k.

Base case $k = 1$. In this case, $f = h_{c,0} e_c$ for some $c \in \overline{\mathbb{Q}}^{\times}$ and the result follows immediately from Lemma 52 because $h_{c,0} e_c$ and $h_{c,0}$ have the same p-Mahler denominator.

Inductive step $k - 1 \to k$. Consider a nonzero generalized p-Mahler series f of the form (71) satisfying $(\mathscr{P} - \mathcal{O}_2)$ (resp. $(\mathscr{P} - \mathcal{O}_3)$) such that $K(f) \leqslant k$. We use the notations (72) and (73). Choose an arbitrary $c_0 \in \mathcal{C}(f)$ and set $j_0 = J(f, c_0)$. We set

$$\widetilde{f} = \sum_{(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}, (c,j) \neq (c_0, j_0)} h_{c,j} e_c \ell^j,$$

so that

$$f = \widetilde{f} + h_{c_0,j_0} e_{c_0} \ell^{j_0}.$$

It follows from Lemma 52 that the p-Mahler denominator $\mathfrak{d} = \mathfrak{d}_{h_{c_0,j_0}}$ of h_{c_0,j_0} has its nonzero roots in $\mathcal{U}$ (resp. $\mathcal{U}_p$). We have

$$\mathfrak{d} h_{c_0,j_0} = \sum_{i=1}^{d} a_i \phi_p^i (h_{c_0,j_0})$$

for some $d \in \mathbb{Z}_{\geqslant 1}$ and some $a_1, \ldots, a_d \in \overline{\mathbb{Q}}[z]$. Therefore, we have

$$\mathfrak{d} f = \mathfrak{d} \widetilde{f} + \mathfrak{d} h_{c_0,j_0} e_{c_0} \ell^{j_0} = \mathfrak{d} \widetilde{f} + \sum_{i=1}^{d} a_i \phi_p^i (h_{c_0,j_0}) e_{c_0} \ell^{j_0}.$$

But,

$$\begin{aligned}
&\sum_{i=1}^{d} a_i \phi_p^i(h_{c_0,j_0}) e_{c_0} \ell^{j_0} \\
= \quad & \sum_{i=1}^{d} c_0^{-i} a_i \phi_p^i(h_{c_0,j_0} e_{c_0} (\ell - i)^{j_0}) \\
= \quad & \sum_{i=0}^{d} \sum_{s=0}^{j_0} c_0^{-i} \binom{j_0}{s} (-i)^s a_i \phi_p^i(h_{c_0,j_0} e_{c_0} \ell^{j_0-s}) \\
= \quad & \sum_{i=1}^{d} c_0^{-i} a_i \phi_p^i(h_{c_0,j_0} e_{c_0} \ell^{j_0}) + \sum_{i=1}^{d} \sum_{s=1}^{j_0} c_0^{-i} \binom{j_0}{s} (-i)^s a_i \phi_p^i(h_{c_0,j_0} e_{c_0} \ell^{j_0-s}) \\
= \quad & \sum_{i=1}^{d} c_0^{-i} a_i \phi_p^i(f - \widetilde{f}) + \sum_{i=1}^{d} \sum_{s=1}^{j_0} c_0^{-i} \binom{j_0}{s} (-i)^s a_i \phi_p^i(h_{c_0,j_0} e_{c_0} \ell^{j_0-s}) \\
= \quad & \sum_{i=1}^{d} a_i c_0^{-i} \phi_p^i(f) + \widetilde{g}
\end{aligned}$$

with

$$\widetilde{g} = -\sum_{i=1}^{d} c_0^{-i} a_i \phi_p^i(\widetilde{f}) + \sum_{i=1}^{d} \sum_{s=1}^{j_0} c_0^{-i} \binom{j_0}{s} (-i)^s a_i \phi_p^i(h_{c_0,j_0} e_{c_0} \ell^{j_0-s}).$$

So,

$$\mathfrak{d} f = \sum_{i=1}^{d} a_i c_0^{-i} \phi_p^i(f) + g \tag{74}$$

where $g = \mathfrak{d}\widetilde{f} + \widetilde{g}$. It is obvious that g satisfies $(\mathscr{P} - \mathcal{O}_2)$ (resp. $(\mathscr{P} - \mathcal{O}_3)$) and $K(g) < K(f) \leqslant k$, so $K(g) \leqslant k - 1$. By induction, Proposition 51 holds true for the generalized p-Mahler series g and it follows from (74) and Lemma 49 that the same is true for f. □

6.5. **Proof of Theorem 11.** We prove Theorem 11 below, after a couple of lemmas. For any $\nu \in \mathbb{Z}_{\geqslant 1}$ relatively prime with p, for any $k \in \mathbb{Z}$, for any element f of $\mathcal{R}$ of the form

$$f = \sum_{(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}} f_{c,j} e_c \ell^j$$

with $f_{c,j} \in \mathscr{H}$, we set

$$\begin{aligned}
[\nu p^k]_* f \quad &= \sum_{(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}} f_{c,j}(z^{\nu p^k}) \phi_p^k(e_c) \phi_p^k(\ell)^j \\
&= \sum_{(c,j) \in \overline{\mathbb{Q}}^{\times} \times \mathbb{Z}_{\geqslant 0}} f_{c,j}(z^{\nu p^k}) c^k e_c (\ell + k)^j.
\end{aligned}$$

Lemma 53. *If f is solution of the equation*

$$a_0(z)y + a_1(z)\phi_p(y) + \cdots + a_d(z)\phi_p^d(y) = 0$$

then $[\nu p^k]_* f$ is solution of the equation

$$a_0(z^{\nu p^k})y + a_1(z^{\nu p^k})\phi_p(y) + \cdots + a_d(z^{\nu p^k})\phi_p^d(y) = 0.$$

Proof. This result follows immediately from the following obvious facts: the map $[\nu p^k]_*$ is additive and, for any $h \in \mathscr{H}$, we have

$$[\nu p^k]_*(hf) = h(z^{\nu p^k})[\nu p^k]_*(f).$$

□

Lemma 54. *Consider a generalized p-Mahler series f, $\nu \in \mathbb{Z}_{\geqslant 1}$ relatively prime with p, $k \in \mathbb{Z}$ and $r \in \{1, 2, 3\}$. Then, f satisfies $(\mathscr{P} - \mathcal{O}_r)$ if and only if $[\nu p^k]_* f$ satisfies $(\mathscr{P} - \mathcal{O}_r)$.*

Proof. Let us first consider the case when f is a Puiseux series. In order to prove the lemma in this case, it is clearly sufficient to prove that if $f \in \mathscr{P}$ satisfies $(\mathcal{O}_r)$ and if $c \in \mathbb{Q}_{>0}$, then $g(z) = f(z^c)$ satisfies $(\mathcal{O}_r)$. Let us prove this. Setting $f = \sum_{\gamma \in \mathbb{Q}} f_\gamma z^\gamma$, we have $g = \sum_{\gamma \in \mathbb{Q}} g_\gamma z^\gamma$ with $g_\gamma = f_{c^{-1}\gamma}$. If $r = 1$, we have $\mathcal{h}(f_\gamma) = \mathcal{O}(H(\gamma))$, so $\mathcal{h}(g_\gamma) = \mathcal{O}(H(c^{-1}\gamma))$. But, $H(c^{-1}\gamma) \leqslant H(c^{-1})H(\gamma)$. So $H(c^{-1}\gamma) = \mathcal{O}(H(\gamma))$ and, hence, $\mathcal{h}(g_\gamma) = \mathcal{O}(H(\gamma))$ so that g satisfies $(\mathcal{O}_1)$ as wanted. The cases $r \in \{2, 3\}$ are similar.

We now come to the general case: we consider a generalized p-Mahler series f, $\nu \in \mathbb{Z}_{\geqslant 1}$ relatively prime with p, $k \in \mathbb{Z}$ and $r \in \{1, 2, 3\}$ and we want to prove that f satisfies $(\mathscr{P} - \mathcal{O}_r)$ if and only if $[\nu p^k]_* f$ satisfies $(\mathscr{P} - \mathcal{O}_r)$. Let

$$f = \sum_{(c,j) \in \overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}, (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in \boldsymbol{\Lambda}_{\mathrm{st}}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j$$

be the standard decomposition of f.

Case $k = 0$. Since $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^\nu) = \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\nu\boldsymbol{a}}(z)$, we have

$$[\nu]_* f = \sum_{(c,j) \in \overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}, (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in \boldsymbol{\Lambda}_{\mathrm{st}}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^\nu) \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\nu\boldsymbol{a}}(z) e_c \ell^j. \tag{75}$$

This is the standard decomposition of $[\nu]_* f$ because ν is relatively prime with p and, hence, the $\nu\boldsymbol{a}$ involved in (75) have entries in $\mathbb{N}_{(p)}$. Then, the following properties are equivalent:

(1) f satisfies $(\mathscr{P} - \mathcal{O}_r)$;
(2) $\forall (c, j) \in \overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}, \forall (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in \boldsymbol{\Lambda}_{\mathrm{st}}$, $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)$ satisfies $(\mathcal{O}_r)$;
(3) $\forall (c, j) \in \overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}, \forall (\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{a}) \in \boldsymbol{\Lambda}_{\mathrm{st}}$, $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^\nu)$ satisfies $(\mathcal{O}_r)$;
(4) $[\nu]_* f$ satisfies $(\mathscr{P} - \mathcal{O}_r)$.

The equivalences between (1) and (2) and between (3) and (4) follow directly from the definition of $(\mathscr{P} - \mathcal{O}_r)$ and the equivalence between (2) and (3) follows from the Puiseux case considered at the very beginning of the proof. This concludes the proof in the case $k=0$.

Case $\nu = 1$ and $k = 1$. Assume first that f satisfies $(\mathscr{P} - \mathcal{O}_r)$. Since $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^p) = \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},p\boldsymbol{a}}(z)$ we have

$$
\begin{aligned}
(76)\qquad [p]_* f &= \sum_{(c,j)\in\overline{\mathbb{Q}}^\times\times\mathbb{Z}_{\geqslant 0},(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in\boldsymbol{\Lambda}_{\mathrm{st}}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^p)\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^p)ce_c(\ell+1)^j \\
&= \sum_{(c,j)\in\overline{\mathbb{Q}}^\times\times\mathbb{Z}_{\geqslant 0},(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in\boldsymbol{\Lambda}} g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)e_c\ell^j
\end{aligned}
$$

where, for each tuple $(c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})$, the Puiseux series $g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)$ is a $\overline{\mathbb{Q}}$-linear combinations of the $f_{c,j',\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}/p}(z^p)$ with $j' \geqslant j$. But, it follows from the Puiseux case considered at the very beginning of the proof that the $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^p)$ satisfy $(\mathcal{O}_r)$. So, the $g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ satisfy $(\mathcal{O}_r)$ as well and, hence, $[p]_*f$ satisfies $(\mathscr{P} - \mathcal{O}_r)$.

Conversely, assume that $[p]_*f$ satisfies $(\mathscr{P} - \mathcal{O}_r)$. Since $\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{\frac{1}{p}}) = \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}/p}(z)$, if we write

$$
[p]_* f = \sum_{(c,j)\in\overline{\mathbb{Q}}^\times\times\mathbb{Z}_{\geqslant 0},(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in\boldsymbol{\Lambda}} g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)e_c\ell^j
$$

where the $g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \mathscr{P}$ satisfy $(\mathcal{O}_r)$, then

$$
\begin{aligned}
(77)\qquad f &= \sum_{(c,j)\in\overline{\mathbb{Q}}^\times\times\mathbb{Z}_{\geqslant 0},(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in\boldsymbol{\Lambda}} g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{\frac{1}{p}})\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{\frac{1}{p}})c^{-1}e_c(\ell-1)^j \\
&= \sum_{(c,j)\in\overline{\mathbb{Q}}^\times\times\mathbb{Z}_{\geqslant 0},(\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in\boldsymbol{\Lambda}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)\xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)e_c\ell^j
\end{aligned}
$$

where, for each tuple $(c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})$, the Puiseux series $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z)$ is a $\overline{\mathbb{Q}}$-linear combinations of the $g_{c,j',\boldsymbol{\alpha},\boldsymbol{\lambda},p\boldsymbol{a}}(z^{\frac{1}{p}})$ with $j' \geqslant j$. But, it follows from the Puiseux case considered at the very beginning of the proof that the $g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{\frac{1}{p}})$ satisfy $(\mathcal{O}_r)$. So, the $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}$ satisfy $(\mathcal{O}_r)$ as well and, hence, f satisfies $(\mathscr{P} - \mathcal{O}_r)$.

General case. The case $k \in \mathbb{Z}_{\geqslant 1}$ follows immediately from the previous particular cases by using the fact that $[\nu p^k]_* = [p]_*^k[\nu]_*$. If $k \in \mathbb{Z}_{\leqslant -1}$, then, using the equality $[\nu]_*f = [p^{-k}]_*[\nu p^k]_*f$ and the cases considered above, we get that f satisfies $(\mathscr{P} - \mathcal{O}_r)$ if and only if $[\nu]_*f$ satisfies $(\mathscr{P} - \mathcal{O}_r)$ if and only if $[p^{-k}]_*[\nu p^k]_*f = [\nu]_*f$ satisfies $(\mathscr{P} - \mathcal{O}_r)$ if and only if $[\nu p^k]_*f$ satisfies $(\mathscr{P} - \mathcal{O}_r)$. This concludes the proof. □

Proof of Theorem 11. Let f be a generalized p-Mahler series satisfying $(\mathscr{P} - \mathcal{O}_r)$ for some $r \in \{1,2,3\}$. We have to prove that the minimal p-Mahler equation of f over $\mathbb{K}_\infty$ has a full basis of generalized p-Mahler series solutions satisfying $(\mathscr{P} - \mathcal{O}_r)$.

Let first consider the case $r = 1$. Theorem 4 ensures that the minimal p-Mahler equation of f over $\mathbb{K}_\infty$ has a full basis of generalized p-Mahler series solutions. Theorem 8 guaranties that any generalized p-Mahler series satisfies $(\mathscr{P} - \mathcal{O}_1)$. This concludes the proof in the case $r = 1$.

We now suppose that $r = 2$ (resp. $r = 3$).

Let us first consider the special case when

$$f = \sum_{(c,j)\in\overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in \bigcup_{t\in\mathbb{Z}_{\geqslant 0}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^\times)^t \times \mathbb{Z}_{>0}^t} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j$$

with $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \overline{\mathbb{Q}}((z))$. Proposition 51 ensures that the p-Mahler denominator $\mathfrak{d}_f \in \overline{\mathbb{Q}}[z]$ of f has its non-zero roots in $\mathcal{U}$ (resp. $\mathcal{U}_p$). Let $a_1, \ldots, a_e \in \overline{\mathbb{Q}}[z]$ be such that f is annihilated by the operator

$$\mathfrak{d}_f - \sum_{i=1}^{e} a_i \phi_p^i. \tag{78}$$

Proposition 48 guaranties that (78) has e $\overline{\mathbb{Q}}$-linearly independent generalized p-Mahler series solutions $f_1, \ldots, f_e$ satisfying $(\mathscr{P} - \mathcal{O}_2)$ (resp. $(\mathscr{P} - \mathcal{O}_3)$). Since the minimal p-Mahler operator of f is a right factor of (78), it has a full set of solutions made of $\overline{\mathbb{Q}}$-linear combinaisons of $f_1, \ldots, f_e$. Such linear combinaisons satisfy $(\mathscr{P} - \mathcal{O}_2)$ (resp. $(\mathscr{P} - \mathcal{O}_3)$), as wanted.

We now come to the general case when

$$f = \sum_{(c,j)\in\overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in \boldsymbol{\Lambda}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j$$

with $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \mathscr{P}$. For any $\nu \in \mathbb{Z}_{\geqslant 1}$ relatively prime with p and any $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$[\nu p^k]_* f = \sum_{(c,j)\in\overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in \boldsymbol{\Lambda}} f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{\nu p^k}) \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\nu p^k \boldsymbol{a}}(z) c^k e_c (\ell + k)^j.$$

From now on, we fix ν and k such that the $f_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}}(z^{\nu p^k})$ belong to $\overline{\mathbb{Q}}((z))$ and the $\nu p^k \boldsymbol{a}$ involved in the previous sum have entries in $\mathbb{Z}_{>0}$. Then, we have

$$[\nu p^k]_* f = \sum_{(c,j)\in\overline{\mathbb{Q}}^\times \times \mathbb{Z}_{\geqslant 0}, (\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a})\in \bigcup_{t\in\mathbb{Z}_{\geqslant 0}} \mathbb{Z}_{\geqslant 0}^t \times (\overline{\mathbb{Q}}^\times)^t \times \mathbb{Z}_{>0}^t} g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \xi_{\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} e_c \ell^j$$

for some $g_{c,j,\boldsymbol{\alpha},\boldsymbol{\lambda},\boldsymbol{a}} \in \overline{\mathbb{Q}}((z))$. Moreover, Lemma 53 guaranties that $[\nu p^k]_* f$ is solution of the p-Mahler equation

$$a_0(z^{\nu p^k}) y + \cdots + a_d(z^{\nu p^k}) \phi_p^d(y) = 0$$

and Lemma 54 ensures that $[\nu p^k]_* f$ satisfies $(\mathscr{P} - \mathcal{O}_2)$ (resp. $(\mathscr{P} - \mathcal{O}_3)$). It follows from the first part of the proof that that the minimal equation over $\mathbb{K}_\infty$ of $[\nu p^k]_* f$ has a full basis of generalized p-Mahler series satisfying $(\mathscr{P}-\mathbb{O}_2)$ (resp. $(\mathscr{P}-\mathbb{O}_3)$). Taking the image of this basis by $[\nu p^k]_*^{-1}$, we get the desired result. □

7. Final remark about the Purity Theorem

Theorem 11 states that the property $(\mathscr{P}-\mathbb{O}_1)$, $(\mathscr{P}-\mathbb{O}_2)$ or $(\mathscr{P}-\mathbb{O}_3)$ of a generalized p-Mahler series is inherited by the other solutions of its minimal equation. The following result shows that this is not true for $(\mathscr{P}-\mathbb{O}_4)$ nor for $(\mathscr{P}-\mathbb{O}_5)$.

Proposition 55. *There exists a p-Mahler power series satisfying $(\mathscr{P}-\mathbb{O}_4)$ and $(\mathscr{P}-\mathbb{O}_5)$ having the following property: its minimal p-Mahler equation has a generalized p-Mahler series solution which does neither satisfy $(\mathscr{P}-\mathbb{O}_4)$ nor $(\mathscr{P}-\mathbb{O}_5)$.*

Proof. Let $p=2$. Consider the following equation

$$y(z)+(z-1)y(z^2)-2zy(z^4)=0. \tag{79}$$

It is well-known that one of its solutions is the generating series $f_{\mathrm{RS}} \in \overline{\mathbb{Q}}[[z]]$ of the Rudin-Shapiro sequence:

$$(a_n)_{n\in\mathbb{Z}_{\geqslant 0}} = 1,1,1,-1,1,1,-1,1,1,1,1,-1,-1,-1,1,\ldots .$$

It's coefficients belong to $\{-1,1\}$, so that it satisfies $(\mathbb{O}_5)$ and, thus, $(\mathscr{P}-\mathbb{O}_4)$ and $(\mathscr{P}-\mathbb{O}_5)$. A study of the Newton polygon of this equation, as in [Roq24], shows that the exponents attached to this equation are 1 and $-\frac{1}{2}$. Thus, it follows from [Roq24] and Theorem 27 that the system associated with (79) has a fundamental matrix of solutions of the form $F_1F_2e_C$, where

$$F_1\in \mathrm{GL}_2(\mathscr{P}),\quad F_2=\begin{pmatrix}1&\xi\\0&1\end{pmatrix},\quad e_C=\begin{pmatrix}1&0\\0&e_{-\frac{1}{2}}\end{pmatrix},$$

with $\xi\in\mathcal{V}_1$. The upper-left entry of F_1 is solution of (79). Hence, up to multiplication by a scalar, we may take it to be f_{RS}. Let $g=(F_1)_{1,2}\in\mathscr{P}$ be the upper-right entry of F_1. Then, a second solution of (79) is the generalized 2-Mahler series $fe_{-\frac{1}{2}}$ where $f=f_{\mathrm{RS}}\xi+g\in\mathscr{H}$.

Using the fact that $\phi_2(e_{-\frac{1}{2}})=-\frac{1}{2}e_{-\frac{1}{2}}$, we obtain that f is solution of the equation

$$y(z)-\frac{1}{2}(z-1)y(z^2)-\frac{1}{2}zy(z^4)=0. \tag{80}$$

Let

$$\chi(z)=\frac{1}{2}\xi_{0,-2,1}(z)=-\sum_{k=1}^{\infty}(-2)^{k-1}z^{-1/2^k}\in\mathscr{H}$$

We claim that we can take $\xi=\chi$. To prove this claim it is sufficient to prove that there exists a Puiseux series $\widetilde{g}$ such that $f_{\mathrm{RS}}\chi+\widetilde{g}$ is solution of

(80). Since $\chi(z^2) = -2\chi(z) - \frac{1}{z}$, it is equivalent to prove that the following equation has a Puiseux solution:

$$y(z) - \frac{1}{2}(z-1)y(z^2) - \frac{1}{2}zy(z^4) = \frac{1}{2z}f_{\mathrm{RS}}(z) - \frac{1}{2z}f_{\mathrm{RS}}(z^4)\,. \tag{81}$$

A study of the Newton polygon associated to (81), as in [CDDM18], shows that this equation has a power series solution. Thus, we can take $\xi = \chi$ and g to be this power series solution of (81). Since the decomposition

$$f = (\frac{1}{2}f_{\mathrm{RS}})\xi_{0,-2,1} + g\xi_{(),(),()}$$

is the standard decomposition of f, to conclude it is sufficient to prove that g do not satisfy $(\mathcal{O}_4)$.

Using (81), it is easily checked that $\mathrm{val}_z\, g = 0$. Looking at the coefficient of z^0 in (81) we obtain that $g_0 = \frac{1}{3}$. Let $g = \sum_{n\geqslant 0} g_n z^n$. Looking at the coefficient of z^1 in (81) and using the fact that $g \in \overline{\mathbb{Q}}[[z]]$ we obtain

$$g_1 - \frac{1}{2}g_0 - \frac{1}{2}g_0 = \frac{1}{2}a_2,$$

where $f_{\mathrm{RS}} = \sum_{n\geqslant 0} a_n z^n$. Thus, $g_1 = \frac{5}{6}$. Now, since g is a power series, looking at the coefficient of z^{2^n} in (81) we obtain,

$$g_{2^n} = \frac{1}{2}a_{2^n+1} - \frac{1}{2}g_{2^{n-1}}$$

Since $a_{2^n+1} = \pm 1$, it follows by induction on n that the 2-adic valuation of g_{2^n} is equal to $n+1$. In particular, $\hbar(g_{2^n}) \geqslant n$. Since $n = \log(H(2^n))/\log(2)$, we have $\hbar(g_\gamma) = \Omega(\log(H(\gamma)))$ and $g(z)$ does not satisfy $(\mathcal{O}_4)$. □

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